THE MATHEMATICAL GAZETTE

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WITH THE CO-OPERATION OF
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AND
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The Mathematical Association.

THE ANNUAL MEETING will be held at the London Day Training College, Southampton Row, London, W.C. 1, on *Thursday, 5th January*, 1928, at 5.30 p.m. (Advanced Section); and on *Friday, 6th January*, 1928, at 10 a.m. and 2.30 p.m. (Ordinary Meeting).

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THE ARITHMETIC OF QUADRATIC NUMBER-FIELDS.

BY PROF. W. E. H. BERWICK, Sc.D.

(A résumé of the contents of this article was read at the Leeds Meeting of the British Association, September 1927.)

1. The ancient Greeks recognised two branches of arithmetic. One of them, the philosophical study, expounded by Euclid in Books VII-IX, developed as a logical science the properties of whole numbers. It was not so much concerned with addition, multiplication and the manipulation of fractions, in which the notation developed in the middle ages has greatly improved the methods of the ancients, as with prime, odd and even numbers, with digressions to triangular, polygonal and perfect numbers. The Greeks were familiar with the facts (1) that when a prime number is a factor of the product of two whole numbers ab it must be a factor of either a or b, and (2) that every whole number can be expressed, in only one way, as a product of prime factors. A late offshoot of this branch of the subject, destined to develop into one division of the modern Theory of Numbers, was considered by Diophantos of Alexandria (English edition by Sir T. L. Heath), who was mainly concerned with finding solutions in rational numbers of indeterminate equations such as

$$yz+a=u^2, \quad zx+a=v^2, \quad xy+a=w^2.$$

The second branch of arithmetic known to the Greeks (λογιστική) was concerned with its applications to measurements, to monetary calculations and to various problems in which the science was used in practical life. No exposition of this is known to have survived the ages.

Soon after the revival of learning there arose a renewed interest in the philosophical study of the science of whole numbers, henceforward to be known as the 'higher arithmetic.' Among its early exponents were Wallis, Mersenne and, above all, Fermat, who, following the custom of their day, announced their discoveries without proof. With one exception the theorems thus given by Fermat, *e.g.* that every prime $4n+1$ is the sum of two integral squares ($89=25+64$, etc.), that every integer is the sum of four or fewer squares, and the well-known 'Fermat's theorem' that $a^p-a\equiv 0 \pmod{p}$, were proved by Euler and others within a hundred years of his death in 1665. The exception is the statement known as Fermat's Last Theorem, that when p is an odd prime the equation

$$x^p+y^p=z^p$$

is impossible in positive integers. Lamé's attempt (1847) to prove this result would have been correct except for the fact that he wrote

$$x^p+y^p=(x+y)(x+y\alpha)\ldots(x+y\alpha^{p-1}), \quad \alpha=\exp.\,2\pi i/p,$$

and assumed that each of the numbers $x+y\alpha^s$ is uniquely expressible as a product of prime factors of the type

$$x_0+x_1\alpha+x_2\alpha^2+\ldots+x_{p-2}\alpha^{p-2},$$

where x_0, etc., are ordinary rational integers.

Although Lamé's argument was invalidated by this assumption, it assisted in stimulating inquiries about the existence of systems of numbers, other than that of rational integers, in which every *integer* (when rightly defined) is uniquely expressible as a product of prime integral factors. At least two such systems were known to Gauss who used

$$a+bi \text{ with } i^2+1=0 \quad \text{and} \quad a+b\omega \text{ with } \omega^2+\omega+1=0,$$

where a, b are rational integers. But the system $a+bi\sqrt{6}$ does not enjoy this property since

$$55=5\,.\,11=(7+i\sqrt{6})(7-i\sqrt{6})=(1+3i\sqrt{6})(1-3i\sqrt{6}),$$

while none of the factors 5, $7+i\sqrt{6}$, etc., are further decomposable into factors. 5 is indecomposable since $5=u^2+6v^2$ is insoluble in integers.

Were $$7+i\sqrt{6}=(p+qi\sqrt{6})(r+si\sqrt{6}),$$

then would $$7-i\sqrt{6}=(p-qi\sqrt{6})(r-si\sqrt{6})$$

and $$55=(p^2+6q^2)(r^2+6s^2),$$

which only admits solutions when one factor on the right is unity. Still it was soon recognised that 5, $7+i\sqrt{6}$ have in common something which also belongs to every integer $5a+(7+i\sqrt{6})b$, and in particular to $1+3i\sqrt{6}$.

2. It is serviceable to work through the proof of the theorems (1) and (2) for ordinary integers and to see how the reasoning fails when applied to numbers $a+b\sqrt{m}$. First, to follow out a plausible argument which appears in some widely used books on algebra. Let p be a prime number (*i.e.* an integer whose only factors are itself and unity), and suppose that p is a factor of the product ab. If p does not divide a, it must, since it is prime and divides the product, be a factor of b. Taking another prime factor of ab and repeating the argument, it follows that ab is uniquely expressible as a product of prime factors. Were this argument correct it would be just as right to say that since $7+i\sqrt{6}$ is only expressible in the form $(a+bi\sqrt{6})(c+di\sqrt{6})$, with a, b, c, d integers, when one factor is unity, and since it divides the product $5\,.\,11$ it must necessarily divide either 5 or 11. This not being the case for

$$\frac{5}{7+i\sqrt{6}}=\frac{1}{11}(7-i\sqrt{6}), \quad \frac{11}{7+i\sqrt{6}}=\frac{1}{5}(7-i\sqrt{6}),$$

the argument is fallacious, and the reader may be interested to discover why.

Euclid's proof of the same theorem is as follows. When $ab=pc$ and p does not divide a, put $$a=x_1p+y_1, \quad 0<y_1<p,$$

also $$p=x_2y_1+y_2, \quad 0<y_2<y_1,$$

and so on, the letters being ordinary integers. Finally

$$y_{n-1}=x_{n+1}y_n+1,$$

and, eliminating the y's in succession,

$$kp+la=1 \text{ with } k, l \text{ integers}.$$

Hence $$b=kbp+lab=p(kb+lc),$$

which is a multiple of p. Repeating this argument every prime factor of ab must divide either a or b, and uniqueness of factorisation follows.

If the same method is applied to numbers of the form $a'+a\sqrt{m}$, where a', a are rational integers and m an integer divisible by no square factor, we can put $$a'+a\sqrt{m}=(p'+p\sqrt{m})(x'+x\sqrt{m})+y'+y\sqrt{m}$$

but cannot generally choose x', x so that $|y'|<|p'|$ and $|y|<|p|$. The proof would be valid if x', x could always be so chosen that

$$|y'^2-my^2|<|p'^2-mp^2|,$$

but this cannot always be done either, *e.g.*

$$a'+a\sqrt{m}=1+\sqrt{-5},\quad p'+p\sqrt{m}=2,$$

$$1+\sqrt{-5}=2(x'+x\sqrt{-5})-(2x'-1)-(2x-1)\sqrt{-5},$$

and there are no rational integers x', x making

$$(2x'-1)^2+5(2x-1)^2<4.$$

Accordingly the Euclidean proof of uniqueness of factorisation into primes, valid for rational integers, generally breaks down when applied to integers of the form $a'+a\sqrt{m}$.

3. Mathematicians of a century ago inquired in what circumstances a system of algebraic numbers such as $a+b\sqrt{m}$ possesses the property of unique factorisation of integers into primes, and sought a means of restoring the quality of unique factorisation into primes to every such system. Much of the pioneering difficulty, to which a further reference is made below (§ 7), in building up a satisfactory theory of algebraic numbers lay in giving correct definitions of integers and of prime integral elements. The problem was solved some fifty years ago by Dedekind, whose theory, so far as it applies to quadratic fields, is outlined here.

Taking m to be a rational integer divisible by no square factor, the *field* $[\sqrt{m}]$ contains all numbers which can be obtained from 1 and $\sqrt{m}$ by any finite combination of the elementary operations of arithmetic: the general member of $[\sqrt{m}]$ is

$$\frac{p+q\sqrt{m}}{r+s\sqrt{m}}\quad\text{or}\quad\frac{(p+q\sqrt{m})(v-s\sqrt{m})}{r^2-ms^2}\quad\text{or}\quad u+v\sqrt{m},$$

where $p, \ldots v$ are rational numbers.

Every field of algebraic numbers contains as a *sub-field* the aggregate of rational numbers [1], of which the integral members are the rational integers

$$0,\ \pm1,\ \pm2,\ \ldots.$$

If A, B are rational integers, the roots of

$$\omega^2+A\omega+B=0$$

are *defined* to be integers too, and called *complex* or *algebraic* integers when they are irrational. The *norm* of ω is defined to be $N(\omega)=\omega\omega'=B$.

Accordingly $u+v\sqrt{m}$, a root of

$$\omega^2-2u\omega+u^2-mv^2=0,$$

is an integer of the field $[\sqrt{m}]$ when $2u$, u^2-mv^2 are rational integers. This necessitates

(1) u, v to be rational integers when $m\equiv2, 3 \pmod 4$,

(2) $2u$, $2v$, $u+v$ to be rational integers when $m\equiv1 \pmod 4$.

So the aggregate of complex integers of $[\sqrt{m}]$ is

$$a+b\sqrt{m}\text{ when }m\equiv2, 3 \pmod 4,$$

and

$$a+b\,\frac{-1+\sqrt{m}}{2}\text{ when }m\equiv1 \pmod 4,$$

a, b being any rational integers.

It will often be convenient to denote a rational integer by R.I. and a complex integer by C.I. The aggregate of R.I.'s possesses the additive and multiplicative properties, *i.e.* $a+b$ and ab are R.I.'s whenever a, b are. It is easily

seen that the aggregate, as above defined, of C.I.'s in a quadratic field is also additive and multiplicative.

4. If $\alpha_1, \alpha_2, \alpha_3, \ldots$ are members, whether integral or not, of a field $[\sqrt{m}]$, the *modulus* $[\alpha_1, \alpha_2, \alpha_3, \ldots]$ is defined to be the aggregate of numbers of the type

$$l_1\alpha_1 + l_2\alpha_2 + l_3\alpha_3 + \ldots$$

for all R.I.'s $l_1, l_2, l_3, \ldots$. Such a modulus possesses an unlimited number of *bases* for $[\alpha_1, \alpha_2, \ldots]$, $[\beta_1, \beta_2, \ldots]$ are identical whenever

$$\left.\begin{aligned}\beta_1 &= p_{11}\alpha_1 + p_{12}\alpha_2 + \ldots,\\ \beta_2 &= p_{21}\alpha_1 + p_{22}\alpha_2 + \ldots,\end{aligned}\right\}\text{ etc.,}\qquad \left.\begin{aligned}\alpha_1 &= q_{11}\beta_1 + q_{12}\beta_2 + \ldots,\\ \alpha_2 &= q_{21}\beta_1 + q_{22}\beta_2 + \ldots,\end{aligned}\right\}\text{ etc.}$$

for R.I. values of the p's and q's : *e.g.* the moduli

$$[15\sqrt{2},\ 3+21\sqrt{2},\ 25+35\sqrt{2}],\quad [15,\ 8+\sqrt{2}]$$

are identical, since

$$8+\sqrt{2} = -13\,.\,15\sqrt{2} + 11(3+21\sqrt{2}) - (25+35\sqrt{2}),\ \text{etc.}$$

The C.I.'s of the field form a modulus known as the *integral basis* $\mathfrak{o}$: a basis of $\mathfrak{o}$ is, after § 3, $[1, \sqrt{m}]$ when $m \equiv 2, 3 \pmod 4$ and $[1, \frac{1}{2}(1+\sqrt{m})]$ when $m \equiv 1 \pmod 4$.

The new entity required in the theory of algebraic numbers passed through various evolutionary stages and was defined in its final form by Dedekind as a particular kind of modulus known as an *ideal*. A modulus $[\alpha_1, \alpha_2, \ldots]$ is said to be an ideal when

(i) all its members are complex integers, and

(ii) $\alpha_1\omega, \alpha_2\omega, \ldots$ are members, where ω is any C.I. of the field. The integral basis $\mathfrak{o} \equiv (1, \alpha)$ is an ideal and, if μ is any C.I. of the field, $(\mu, \mu\alpha)$ is another. Condition (ii) in the definition of an ideal is satisfied provided

$$\alpha_i\alpha = l_{1i}\alpha_1 + l_{2i}\alpha_2 + \ldots \text{ with every } l_{ji} \text{ a R.I.}$$

Every ideal has a two-member basis, for if

$$\alpha_i = b_i + c_i\alpha, \quad \text{and} \quad c = dv(c_1, c_2, \ldots),^*$$

there exist R.I.'s $k_1, k_2, \ldots$ such that

$$k_1c_1 + k_2c_2 + \ldots = c,$$
$$k_1\alpha_1 + k_2\alpha_2 + \ldots = b + c\alpha.$$

The R.I.

$$b_i + c_i\alpha - \frac{c_i}{c}(b + c\alpha) \equiv a_i$$

is a member of the ideal, and if $a = dv(a_1, a_2, \ldots)$, the ideal

$$\mathfrak{a} = (\alpha_1, \alpha_2, \ldots) = (a, b + c\alpha).$$

(Any modulus, whether an ideal or not, can be reduced to a two-member basis in the same way.)

We next examine in what circumstances the modulus $[a, b+c\alpha]$ is an ideal $(a, b+c\alpha)$. It is necessary and sufficient that $a\alpha$, $(b+c\alpha)\alpha$ should both belong to the modulus.

First when $m \equiv 2, 3 \pmod 4$, $\alpha = \sqrt{m}$,

$$a\alpha = \frac{a}{c}(b + c\alpha) - \frac{ab}{c},$$

$$(b + c\alpha)\alpha = \frac{b}{c}(b + c\alpha) - \frac{1}{c}(b^2 - mc^2),$$

---

* The greatest common measure of the rational integers $c_1, c_2, \ldots$, taken positively, is denoted by $dv(c_1, c_2, \ldots)$.

and the conditions are

$$a\equiv 0 \pmod{c}, \quad b\equiv 0 \pmod{c}, \quad b^2-mc^2\equiv 0 \pmod{ac}.$$

But when $m\equiv 1 \pmod 4$, $\alpha=\frac{1}{2}(1+\sqrt{m})$,

$$(b+c\alpha)\alpha=\frac{b+c}{c}(b+c\alpha)-\frac{1}{c}[b^2+bc+\tfrac{1}{4}(1-m)c^2],$$

and the modulus is an ideal, provided

$$a\equiv b\equiv 0 \pmod{c}, \quad b^2+bc+\tfrac{1}{4}(1-m)\,c^2\equiv 0 \pmod{ac}.$$

Thus

$$(3, \sqrt{15}), (4, 3+\sqrt{17}), (5+10i, 6-3i, 7-6i), (24, 7+i\sqrt{47}), (31, 23+\sqrt{2})$$

are ideals, whereas

$$[3, i\sqrt{15}], [3, 1+2\sqrt{10}], [8, 3+\sqrt{17}], [48, 1+i\sqrt{47}] \text{ are not.}$$

The *norm* of the ideal $\mathfrak{a}\equiv(a, b+c\alpha)$, when a, c are positive, is $N(\mathfrak{a})=ac$.

5. The product of two ideals,

$$\mathfrak{a}\equiv(\alpha_1, \alpha_2, \ldots)\equiv(a, b+c\alpha), \quad \mathfrak{b}\equiv(\beta_1, \beta_2, \ldots)\equiv(a', b'+c'\alpha)$$

is the aggregate

$$\mathfrak{ab}\equiv(aa', a(b'+c'\alpha), a'(b+c\alpha), (b+c\alpha)(b'+c'\alpha)),$$

which can be reduced to a two-member basis $(A, B+C\alpha)$ and is also an ideal, for the conditions (i), (ii) are immediately seen to be satisfied.

An ideal $\mathfrak{b}$ is divisible by another ideal $\mathfrak{a}$ when every member of $\mathfrak{b}$ is also a member of $\mathfrak{a}$. The ideal $\mathfrak{p}$ is prime when it is divisible by no ideal except itself and $\mathfrak{o}$. Salient points in Dedekind's theory are

I. $N(\mathfrak{ab})=N(\mathfrak{a})\,N(\mathfrak{b})$,

II. when $\mathfrak{b}$ is divisible by $\mathfrak{a}$, according to the above definition, there exists an ideal $\mathfrak{c}$ such that $\mathfrak{b}=\mathfrak{ac}$.

III. every ideal can be uniquely expressed as a product of prime ideal factors.

It is not the present purpose to prove these results but merely to illustrate their numerical application.

6. We proceed to distinguish three types of prime ideal. First, when m is a quadratic residue of p, an odd prime which does not divide m, there exists an odd integer c such that $c^2-m\equiv 0 \pmod p$; the ideal $(p, p\alpha)$ then

$$=(p, c+\sqrt{m})(p, -c+\sqrt{m}) \text{ when } m\equiv 2, 3 \pmod 4,$$

and

$$=(p, \tfrac{1}{2}(c-1)+\alpha)(p, -\tfrac{1}{2}(c+1)+\alpha) \text{ when } m\equiv 1 \pmod 4.$$

These prime ideals are said to be of the first degree.

Of the integers $1, 2, \ldots p-1$ half are quadratic residues of p and the other half not (*e.g.* when $p=7$,

$$1^2\equiv 6^2\equiv 1, \quad 2^2\equiv 5^2\equiv 4, \quad 3^2\equiv 4^2\equiv 2$$

and 1, 4, 2 are the quadratic residues of 7). When m is a quadratic non-residue of p, the congruence

$$c^2-m\equiv 0 \pmod p$$

is insoluble and $(p, p\alpha)$ is a prime ideal of the second degree, its norm being p^2.

Thirdly, when p is a factor of m, $(p, p\alpha)$ is the square of a prime ideal: in fact,

$$(p, p\sqrt{m})=(p, \sqrt{m})^2 \text{ when } m\equiv 2, 3 \pmod 4,$$

$$(p, p\alpha)=(p, \tfrac{1}{2}(p+\sqrt{m}))^2 \text{ when } m\equiv 1 \pmod 4.$$

Corresponding results under $p=2$ are:

$$(2, 2\sqrt{m})=(2, \sqrt{m})^2 \text{ when } m\equiv 2 \pmod 4,$$

$$(2, 2\sqrt{m})=(2, 1+\sqrt{m})^2 \text{ when } m\equiv 3 \pmod 4,$$

$$(2, 2a)=(2, a)(2, 1+a) \text{ when } m\equiv 1 \pmod 8,$$

while $(2, 2a)$ is a prime ideal when $m\equiv 5 \pmod 8$. The first two of these results are easily proved. When $m\equiv 1 \pmod 8$ both $(2, a)$ and $(2, 1+a)$ are ideals by definition, their products being

$$(4, 2a, 2+2a, \tfrac{1}{4}(m-1)+2a)=(2, 2a),$$

as stated. But when $m\equiv 5 \pmod 8$, neither of the moduli $[2, a]$, $[2, 1+a]$ satisfies the conditions of an ideal, and no ideal of norm 2 exists. Hence in this case $(2, 2a)$ is prime.

Numerical examples of multiplication are easily constructed: here we take the fields $[\sqrt{-5}]$ and $[\sqrt{-23}]$.

When $\theta^2+5=0$, $\mathfrak{o}=(1, \theta)$,

$$(2, 1+\theta)(7, 3+\theta)=(14, 3+\theta),$$

$$(3, 1+\theta)^3=(27, 7+\theta),$$

$$(7, 3+\theta)(23, 15+\theta)=(161, 38+\theta).$$

And when $a^2-a+6=0$,

$$a=\tfrac{1}{2}(1+\sqrt{-23}), \quad \mathfrak{o}=(1, a), \quad (2, a)^3=(8, a-2),$$

$$(3, a-1)^4=(81, a-13), \quad (2, a)^2(3, a-1)(13, a-5)=(156, a-70).$$

(The relation $N(\mathfrak{a})\,N(\mathfrak{b})=N(\mathfrak{ab})$ provides a useful check on the accuracy of every multiplication.)

Generally, when $(a, b+ca)$ is an ideal,

$$(f, fa)(a, b+ca)=(af, bf+cfa).$$

7. The factorisation of an ideal $(a, b+ca)$ can also be carried out with little difficulty. When $c\neq 1$, put

$$a=ca', \quad b=cb',$$

then

$$(a, b+ca)=(c, ca)(a', b'+a),$$

and, if $c=pp'p''\ldots$,

$$(c, ca)=(p, pa)(p', p'a)(p'', p''a)\ldots.$$

The ideal (p, pa) is a prime ideal of the second degree when m is a quadratic non-residue of p; otherwise it is the product of two prime ideals of the first degree, as noted in § 6.

If q is a prime factor of a', the ideal $\mathfrak{a}'\equiv(a', b'+a)$ is divisible by $\mathfrak{q}\equiv(q, b'+a)$. To find the quotient we multiply $\mathfrak{a}'$ by the conjugate ideal $\mathfrak{q}'$ and then remove $\mathfrak{qq}'=(q, qa)$. Thus $(7, 5+\sqrt{95})$ is a factor of $(35, 5+\sqrt{95})$.

$$(35, 5+\sqrt{95})(7, 5-\sqrt{95})$$

$$=(245, 175-35\sqrt{95}, 35+7\sqrt{95}, 70)=(35, 7\sqrt{95})$$

$$=(7, 7\sqrt{95})(5, \sqrt{95})=(7, 5+\sqrt{95})(7, 5-\sqrt{95})(5, \sqrt{95}),$$

whence $(35, 5+\sqrt{95})=(7, 5+\sqrt{95})\,(5, \sqrt{95})$.

[At this stage it is fitting to mention a difficulty of the early investigators. If the integral basis of $[\sqrt{5}]$ is taken to be $(1, \sqrt{5})$ and $(2, 1+\sqrt{5})$ to be an ideal factor of 2,

$$(2, 1+\sqrt{5})^2=(4, 2+2\sqrt{5}, 6+2\sqrt{5})=(4, 2+2\sqrt{5}),$$

and the relation $N(\mathfrak{a})\,N(\mathfrak{b})=N(\mathfrak{ab})$ appears to be invalid. Except for difficulties of this type the whole problem of the ideal theory would have been solved by Zolotareff or others long before Dedekind's day. The explanation of the above fallacy is the fact that, in the field $[\sqrt{5}]$, $(2, 1+\sqrt{5})$ is a prime ideal of the second degree.]

The discriminant of the field is $D \equiv (\alpha - \alpha')^2$ which $= m$ or $4m$ according as $m \equiv 1$ or $\equiv 2, 3 \pmod 4$. It has been seen that

IV. prime factors of D are the only primes divisible by the square of a prime ideal.

8. An ideal $(a, b + c\alpha)$ is said to be *principal* when it is the aggregate of C.I. multiples of a C.I. μ of the field: such an ideal is denoted by (μ). For example,

$$(11, 3 + \sqrt{-2}) = (3 + \sqrt{-2}), \quad (61, 19 + \sqrt{-5}) = (4 - 3\sqrt{-5}).$$

Two ideals $\mathfrak{a}$, $\mathfrak{b}$ are *equivalent*, denoted by $\mathfrak{a} \sim \mathfrak{b}$, when there exist two C.I.'s α, β such that $(\alpha)\mathfrak{a} = (\beta)\mathfrak{b}$. An alternative definition of equivalence, amounting to the same thing, is that $\mathfrak{a} \sim \mathfrak{b}$ when there exists an ideal $\mathfrak{k}$ such that $\mathfrak{a}\mathfrak{k}$, $\mathfrak{b}\mathfrak{k}$ are both principal. Equivalent ideals belong to the same *class*, and fundamental theorems which will not be proved here are

V. that every class contains at least one ideal $\mathfrak{a}$ such that $N(\mathfrak{a}) < |\sqrt{D}|$, and

VI. that the number of distinct classes for a given field is finite.

In dealing with a particular field the following problems present themselves:

(*A*) Given two ideals $\mathfrak{a}$, $\mathfrak{b}$, to determine whether they are equivalent or not;

(*B*) When $\mathfrak{a} \sim \mathfrak{b}$, to connect these ideals by an equation $(\alpha)\mathfrak{a} = (\beta)\mathfrak{b}$, and, in particular, to express any principal ideal in the form (μ);

(*C*) To obtain the number of distinct classes of ideals in a field $[\sqrt{m}]$.

When examining these questions it is necessary to take account of the sign of m which will be assumed negative in §§ 9-11 and positive in § 12.

9. If k is any R.I., the ideal

$$\mathfrak{a} = (a, b' + \alpha) = (a, b' + ka + \alpha)$$

and k can be uniquely chosen so that

$$-\tfrac{1}{2}a \leqslant \text{real part of } b' + ka + \alpha < \tfrac{1}{2}a.$$

Putting $b = ka + b'$, the ideal

$$\mathfrak{a} = (a, b + \alpha) \text{ and, by definition, } N(b + \alpha) = aa_1.$$

Whenever $a > |\sqrt{D}|$, and frequently when $a < |\sqrt{D}|$, $a_1 < a$. Hence, from the fundamental identity

$$a_1(a, b + \alpha) = (a_1 a, a_1 b + a_1 \alpha) = (b + \alpha)(a_1, b + \alpha'),$$

the ideal $\mathfrak{a}$ is linked up with an equivalent ideal $\mathfrak{a}_1$, generally of smaller norm. The process can be repeated, thus

$$a_1(a, b + \alpha) = (b + \alpha)(a_1, b_1 + \alpha),$$

$$a_2(a_1, b_1 + \alpha) = (b_1 + \alpha)(a_2, b_2 + \alpha), \text{ etc.},$$

with $a > a_1 > a_2 >$ etc., and, since the a's are all positive integers, a stage will be reached when $a_n \leqslant a_{n+1}$. The ideal $(a_n, b_n + \alpha)$ is then said to be *reduced*: $a_n < |\sqrt{D}|$ always, and when $(a, b + \alpha)$ is principal $a_n = 1$.

As an example of reduction we have

$$26 \,.\, (601, 125 + i) = (125 + i)(26, 5 + i),$$

$$1 \,.\, (26, 5 + i) = (5 + i)(1, i),$$

whence, multiplying right and left,

$$(601, 125 + i) = \frac{(125 + i)(5 + i)}{26} = (24 + 5i).$$

Similarly, $(601, -125 + i) = (24 - 5i)$, hence

$$(601) = (601, 125 + i)(601, -125 + i) = (24 + 5i)(24 - 5i),$$

$$i.e. \quad 601 = 24^2 + 5^2.$$

As another example we determine whether the two ideals

$$\mathfrak{a}=(71,\ \alpha-20),\quad \mathfrak{b}=(26,\ \alpha-17)$$

in the field $[\sqrt{-23}]$ given by $\alpha^2+\alpha+6=0$

are equivalent or not.

Reducing them,

$$6(71,\ \alpha-20)=(\alpha-20)\,(6,\ \alpha'-20)=(\alpha-20)(6,\ \alpha+3),$$
$$2(6,\ \alpha+3)=(\alpha+3)(2,\ \alpha);$$
$$12(26,\ \alpha-17)=(\alpha-17)(12,\ \alpha'-17)=(\alpha-17)(12,\ \alpha+6),$$
$$3(12,\ \alpha+6)=(\alpha+6)(3,\ \alpha+1),$$
$$2(3,\ \alpha+1)=(\alpha+1)(2,\ \alpha).$$

Thus both ideals are equivalent to the reduced ideal $(2,\ \alpha)$: in fact,

$$(71,\ \alpha-20)=\frac{(\alpha-20)(\alpha+3)}{12}(2,\ \alpha)=\frac{3\alpha+11}{2}(2,\ \alpha),$$

$$(26,\ \alpha-17)=\frac{(\alpha-17)(\alpha+6)(\alpha+1)}{12\,.\,3\,.\,2}(2,\ \alpha)=\frac{3\alpha+1}{2}(2,\ \alpha),$$

so $$(3\alpha+11)(26,\ \alpha-17)=(3\alpha+1)(71,\ \alpha-20).$$

10. We next search for the reduced ideals in the field $[\sqrt{-23}]$ defined as in § 9. Since $|\sqrt{D}|<5$ it is only necessary to examine the possibilities

$$\mathfrak{o}\equiv(1,\ \alpha),\ \mathfrak{p}_2=(2,\ \alpha),\ \mathfrak{q}_2=(2,\ \alpha+1),$$
$$\mathfrak{p}_3=(3,\ \alpha),\ \mathfrak{q}_3=(3,\ \alpha+1),\ \mathfrak{p}_2^2=(4,\ \alpha+2),\ \mathfrak{q}_2^2=(4,\ \alpha-1),$$

for $\mathfrak{p}_2\mathfrak{q}_2=(2)\sim\mathfrak{o}$. Reducing these we find

$$\mathfrak{p}_3\sim\mathfrak{q}_2,\quad \mathfrak{q}_3\sim\mathfrak{p}_2,\quad \mathfrak{p}_2^2\sim\mathfrak{q}_2,\quad \mathfrak{q}_2^2\sim\mathfrak{p}_2,$$

hence there are only three non-equivalent reduced ideals $\mathfrak{o}$, $\mathfrak{p}_2$, $\mathfrak{q}_2$ and the class-number $h=3$. It is convenient to assign the class-symbols C, C^2, $C^3\equiv1$ to the respective ideals $\mathfrak{p}_2$, $\mathfrak{q}_2$, $\mathfrak{o}$. Every ideal of the field belongs to one of the three classes C, C^2, 1 and an application of the reduction process serves to determine which. Also, if $\mathfrak{a}\sim C^r$, $\mathfrak{b}\sim C^s$, the product $\mathfrak{ab}\sim C^{r+s}$.

If $\mathfrak{a}$ is any ideal in $[\alpha]$, $\mathfrak{a}^3$ belongs to the principal class,

i.e. $$\mathfrak{a}^3=(u+v\alpha),\quad N(\mathfrak{a}^3)=a^3=u^3-uv+6v^2.$$

For example, $$\mathfrak{a}=(13,\ \alpha-4)\sim C^2,$$
$$\mathfrak{p}_2\mathfrak{a}=(\alpha-4),\quad \mathfrak{q}_2^2\mathfrak{a}=(3\alpha+1),$$
$$4\mathfrak{a}^3=(\mathfrak{p}_2\mathfrak{a})^2(\mathfrak{q}_2^2\mathfrak{a})=(\alpha-4)^2(3\alpha+1)=4(12\alpha+43),$$

whence $$\mathfrak{a}^3=(12\alpha+43)=37+6\sqrt{-23},$$

and $$13^3=37^2+23\,.\,6^2.$$

11. In a few fields ($-m=1,\ 2,\ 3,\ 7,\ 11,\ 19,\ 43,\ 67,\ 163$) all the ideals are principal and factorisation of complex integers into prime factors can be carried out uniquely, just as with rational integers. For example,

$$\mu=59+9i\sqrt{7},\quad N(\mu)=4048=2^4\,.\,11\,.\,23.$$

Putting $\alpha^2+\alpha+2=0$, $\mu=68+18\alpha$,

$$(2)=(2,\ \alpha)(2,\ \alpha+1)=-\alpha(\alpha+1)=\mathfrak{p}_2\mathfrak{q}_2,$$
$$(11)=(11,\ 5+\alpha)(11,\ -4+\alpha)=(2-i\sqrt{7})(2+i\sqrt{7}),$$
$$(23)=(23.\ 10+\alpha)(23,\ -9+\alpha)=(4-i\sqrt{7})\,(4+i\sqrt{7}).$$

$\frac{1}{2}\mu$ is a C.I., a member of $\mathfrak{p}_2$ but not of $\mathfrak{q}_2$, so $\mu=2\alpha^2\gamma$ where $N(\gamma)=11\,.\,23$. Since also μ is a member of the ideals $(2-i\sqrt{7})$ and $(4+i\sqrt{7})$ but not of $(2+i\sqrt{7})$ or $(4-i\sqrt{7})$, the prime factors are necessarily given by

$$\mu=\pm\alpha^3(\alpha+1)(2\alpha-1)(2\alpha+5),$$

and, on multiplication, the sign is found to be negative.

An exceptional feature occurs in the fields $[i]$ and $[i\sqrt{3}]$ which possess C.I.'s of unit norm. Such C.I.'s are called *units*.

In $[i]$ the units are ± 1, $\pm i$ and if μ is any C.I. prime to $1+i$, so that $N(\mu)$ is odd, just one of the four C.I.'s,

$$\mu, \quad -\mu, \quad i\mu, \quad -i\mu,$$

is congruent to 1 (mod $2+2i$). Such a C.I. is said to be *primary*, and the product of two primary numbers is easily seen to be primary. A positive prime p, $\equiv 1 \pmod 4$, is the product of primary complex prime factors, and any integer θ of the field admits the unique factorisation

$$\theta = i^s(1+i)^t \varpi_1 \varpi_2 \dots \varpi_r$$

with ϖ_i primary.

Again, in $[i\sqrt{3}]$, with $\omega = \frac{1}{2}(-1+i\sqrt{3})$, the units are

$$1, \quad -1, \quad \omega, \quad -\omega, \quad \omega^2, \quad -\omega^2,$$

and a primary C.I. $\equiv 1 \pmod 3$. Every positive prime $6n+1$ is the product of two primary factors in the field, and, if μ is prime to $1-\omega$, just one of the six C.I.'s,

$$\mu, \quad -\mu, \quad \omega\mu, \quad -\omega\mu, \quad \omega^2\mu, \quad -\omega^2\mu,$$

is primary. Any C.I. of the field is uniquely expressible in the form

$$\pm\omega^s(1-\omega)^t \varpi_1 \varpi_2 \dots \varpi_r$$

with ϖ_i primary. For example,

$$20 + i\sqrt{3} = -\omega(1-3\omega)(1-6\omega).$$

The number of negative values of m for which the class-number has an assigned value is probably finite, but this has not actually been proved. Corresponding to $h = 1, 2, 3$, the greatest known values of $-m$ are 163, 427, 907 respectively.

12. Passing to positive values of m an infinite number of complex units exists in every field. A unit is a C.I. whose reciprocal is integral and hence a root of

$$\epsilon^2 + A\epsilon \pm 1 = 0.$$

To reduce an ideal $(a, b'+\sqrt{m})$, when $m \equiv 2, 3 \pmod 4$, we first replace b' by $b \equiv b' + ka$, k being so chosen that

$$0 \leqslant b < a \quad \text{when} \quad a > 2\sqrt{m}$$

and

$$0 < \sqrt{m} - b < a \quad \text{when} \quad a < 2\sqrt{m}.$$

Putting $aa_1 = b^2 - m$ in the first case and $aa_1 = m - b^2$ in the second, the identity

$$a_1(a, b+\sqrt{m}) = (b+\sqrt{m})(a_1, -b_1+\sqrt{m})$$

connects $(a, b+\sqrt{m})$ with an equivalent ideal and the process can be repeated.* After a certain stage the sequence of equivalent ideals becomes periodic and the reduced ideals are the ones belonging to the recurring cycle. For example,

$$(59, 45+\sqrt{19}) \sim (34, 23+\sqrt{19}) \sim (15, 7+\sqrt{19})$$
$$\sim (2, 3+\sqrt{19}) \sim (5, 2+\sqrt{19}) \sim (3, 4+\sqrt{19}) \sim (1, 4+\sqrt{19})$$
$$\sim (3, 2+\sqrt{19}) \sim (5, 3+\sqrt{19}) \sim (2, 3+\sqrt{19}),$$

after which the sequence recurs. The reduced ideals, from $(2, 3+\sqrt{19})$ to $(5, 3+\sqrt{19})$, are all principal, since $(1, 4+\sqrt{19})$ belongs to the set.

Multiplying, right and left, the equations, from the successive reduced ideals only,

$$5(2, 3+\sqrt{19}) = (3+\sqrt{19})(5, 2+\sqrt{19}), \text{ etc.},$$

the two principal ideal $(2 \cdot 5 \cdot 3 \cdot 1 \cdot 3 \cdot 5)$ and

$$(\gamma) \equiv (3+\sqrt{19})(2+\sqrt{19})(4+\sqrt{19})(4+\sqrt{19})(2+\sqrt{19})(3+\sqrt{19})$$

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* There is a slight modification when $m \equiv 1 \pmod 4$, $\alpha = -\frac{1}{2}+\frac{1}{2}\sqrt{m}$, $\alpha' = -\frac{1}{2}-\frac{1}{2}\sqrt{m}$, k being so chosen that $0 < b - \frac{1}{2} < a$ when $a > \sqrt{m}$, $0 < \sqrt{m} - b - \frac{1}{2} < a$ when $a < \sqrt{m}$.

Putting $aa_1 = |N(b+\alpha)|$ we have

$$a_1(a, b+\alpha) = (b+\alpha)(a_1, b+\alpha')$$

and continue as before.

are equal, and, since $3+\sqrt{19}>2$, etc., both $\epsilon\equiv\gamma/2\,.\,5\,.\,3\,.\,3\,.\,5$ and its reciprocal are C.I.'s. This number $\epsilon(=170+39\sqrt{19})$ is the *primitive unit* of the field and all units are included in the formula $\pm\epsilon^n$.

To express $(59,\ 45+\sqrt{19})$ in the form (μ) we have

$$34(59,\ 45+\sqrt{19})=(45+\sqrt{19})(34,\ 23+\sqrt{19}),\ \text{etc.},$$

whence

$$\mu=\frac{(45+\sqrt{19})(23+\sqrt{19})(7+\sqrt{19})(3+\sqrt{19})(2+\sqrt{19})(4+\sqrt{19})}{34\,.\,15\,.\,2\,.\,5\,.\,3}$$

$$=196+45\sqrt{19},$$

and, as confirmation,

$$N(\mu)=-59,\quad 45+\sqrt{19}=(196+45\sqrt{19})(-135+31\sqrt{19}),$$

whence $(59,\ 45+\sqrt{19})$

$$=(196+45\sqrt{19})(45\sqrt{19}-196,\ 31\sqrt{19}-135)=(196+45\sqrt{19})\mathfrak{o}.$$

The general solution of $19y^2-x^2=59$ is given by $\sqrt{19}\,.\,y+x=\mu\epsilon^n$.

If $(a,\ b+\alpha)$ is an ideal in the principal class a or $-a$ is a norm according as the number of ideals for which $N(b_i+\alpha)<0$ passed through in reaching (1) is even or odd. (Passing from $(59,\ 45+\sqrt{19})$ to $(1,\ 4+\sqrt{19})$ there are three such ideals, the fourth, fifth and sixth, hence $N(\mu)=-59$.) When the cycle contains an odd number of reduced ideals $N(\epsilon)=-1$ and

$$N(\mu)=a,\quad N(\mu)=-a$$

are both possible.

There is one recurring cycle in each class, and every cycle generates the same primitive unit.

When $m=105$ the cycles are

$$C_0^2\sim(1,\ 5+\alpha)\sim(6,\ 2+\alpha)\sim(4,\ 3+\alpha)\sim(5,\ 3+\alpha)$$
$$\sim(4,\ 2+\alpha)\sim(6,\ 5+\alpha),$$
$$C_0\sim(3,\ 5+\alpha)\sim(2,\ 4+\alpha)\sim(7,\ 4+\alpha)\sim(2,\ 5+\alpha),$$

and

$$\epsilon=45+8\alpha=41+4\sqrt{105}.$$

13. When $N(b+\alpha)=aa_1$, this number is a quadratic residue of p, any odd prime factor of m, and the ideals $(a,\ b+\alpha)$, $(a_1,\ b+\alpha')$ are equivalent. Hence a, a_1, when prime to p, are both quadratic residues or else both quadratic non-residues of p. Since any two ideals in the same class can be connected, by one or more steps, in this way, it follows that the norm (when prime to p) of every ideal in a class has the same quadratic character with respect to p. This defines the character of the class, + when the norms are quadratic residues, − when they are non-residues of p.

Multiplying two ideals in the classes C, C', it is easily seen that the character with respect to p of the class CC' is the product of the characters of C and C'. The classes having the character + for every odd prime factor of m form the *principal genus*. For most values of m (including all small values) these classes can be arranged to form a single period

$$1,\ C,\ C^2,\ \ldots\ C^{h-1}.$$

In the most general case take C to be a class (of the principal genus) developing the longest period $1,\ C,\ \ldots\ C^{k-1}$ of k classes. Among those that remain let $F^{k'}=1$ with k' as great as possible, and so on. Then all the classes in the principal genus are

$$C^xF^yG^z\ldots,\quad 0\leqslant x<k,\quad 0\leqslant y<k',\quad 0\leqslant z<k'',$$

with $kk'k''\ldots=h$. No negative value of m requiring more than two k's is known, nor any positive value requiring more than one.

Among the classes not contained in the principal genus there is always a set of r, $C_1, C_2, \ldots C_r$, say, such that

(i) $C_1^2 \sim C_2^2 \sim \ldots \sim C_r^2 \sim (1)$,
(ii) these classes are not connected by any relation

$$C_1^{x_1} C_2^{x_2} \ldots C_r^{x_r} = 1$$

unless all the x's are even. The whole aggregate of classes is then

$$C_1^{x_1} C_2^{x_2} \ldots C_r^{x_r} C^x F^y G^z, \quad x_i = 0 \text{ or } 1,$$

and the class-number is $2^r \cdot h$, or better $\Pi^r \cdot h$.

As an example of the arrangement we take the field $\theta^2 + 74 = 0$ and represent each of the ten classes by the unique reduced ideal therein.

+	−
$C \sim (3, 1+\theta)$	$CC_1 \sim (6, -2+\theta)$
$C^2 \sim (9, 4+\theta)$	$C^2C_1 \sim (5, 1+\theta)$
$C^3 \sim (9, -4+\theta)$	$C^3C_1 \sim (5, -1+\theta)$
$C^4 \sim (3, -1+\theta)$	$C^4C_1 \sim (6, 2+\theta)$
$C^5 \sim (1, \theta)$	$C_1 \sim (2, \theta)$

14. Dedekind's theory applies also to the field of algebraic numbers defined by any irreducible equation $a(\theta) = 0$ of degree n. In the field $[\theta]$ the C.I.'s form a modulus of order n,

$$h_0\omega_0 + h_1\omega_1 + \qquad + h_{n-1}\omega_{n-1}$$

where $\omega_0, \omega_1, \ldots \omega_{n-1}$ is a suitably chosen set of C.I.'s and the h's are R.I.'s. If

$$\omega_i\alpha_j = k_{ij0}\alpha_0 + k_{ij1}\alpha_1 + \ldots k_{ijn-1}\alpha_{n-1}$$

with k_{ijl} a R.I., $(\alpha_0, \alpha_1, \ldots \alpha_{n-1})$ is a basis of an ideal in the field. The theorems I-VI are true verbatim for a field of any degree.

If the equation $a(\theta) = 0$ has r real roots and $2s$ complex ones, the field contains a set of $t \equiv r+s-1$ primitive units

$$\epsilon_1, \epsilon_2, \ldots \epsilon_t,$$

connected by no relation of the type

$$\epsilon_1^{y_1} \epsilon_2^{y_2} \ldots \epsilon_t^{y_t} = 1$$

unless every $y_i = 0$, and such that every unit is a product of integral powers of the ϵ's.

The class-symbols are commutative and define an Abelian group. Except in quadratic fields, the theory of which has been sketched above, and cubic fields with negative discriminants, there is no systematic method of deciding whether two ideals are equivalent or of reducing a principal ideal to the form (μ). An analytical formula, involving the Dedekind ζ-function, connects the number of classes of ideals with a primitive set of units, but, to a true arithmetician, this is not the correct way to find either the units or the class-number.

15. Dedekind's theory of algebraic numbers is to be found in the last supplement of the fourth edition of Dirichlet's *Zahlentheorie* (1875). A full account of the investigations to which it led in the next twenty years is given in Hilbert's report, *Jahresber. d. Deut. Math. Ver. IV* (1897). This has been brought up to date by Dickson and others, Bull. Nat. Research Council, Feb. 1923 (Washington)—Algebraic Numbers.

An extensive account of Quadratic Fields is given in Sommer's *Vorlesungen ü. Zahlentheorie* (Teubner, 1907). Reid's *Algebraic Numbers* (Macmillan, 1911) contains a useful, and easy, introduction to the subject. W. E. H. BERWICK.

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NON-COMMUTATIVE ALGEBRA.*

By Prof. H. W. Turnbull, M.A.

The whole of ordinary algebra, dealing with real or complex numbers a, b, c, is based on three fundamental laws which are usually called:

I. The Associative Law,

II. The Distributive Law,

III. The Commutative Law.

Each of these leads to two primitive facts concerning ordinary addition and multiplication, as follows:

I. (i) $(a+b)+c=a+(b+c)$, I. (ii) $(a\times b)\times c=a\times(b\times c)$,

II. (i) $a\times(b+c)=(a\times b)+(a\times c)$, II. (ii) $(a+b)\times c=(a\times c)+(b\times c)$,

III. (i) $a+b=b+a$, III. (ii) $a\times b=b\times a$.

From these six propositions, as others have proved, the whole of the algebra of ordinary numbers can be deduced, which alone is reason enough to prompt a scrutiny into them and to discover their relative importance. By far the most interesting is the last, the commutative law of multiplication: and there can be little doubt that whoever first discovered it was led thereto by the orderly arrangement of n things in r rows or of r things in n rows.

Now these six propositions are not entirely independent, for II. (ii) can be deduced from II. (i) and III. (ii), since

$$\begin{aligned}(a+b)\times c&=c\times(a+b)\\&=c\times a+c\times b\\&=a\times c+b\times c.\end{aligned}$$

But curiously enough the converse is not true, for we cannot deduce the law $a\times b=b\times a$ by assuming the truth of the other five.

As a result of considerable experience since Hamilton invented quaternions and Grassmann their generalized forms, called extensive magnitudes, now nearly a hundred years ago, it has come to be recognized that there is a profound cleavage between the commutative law of multiplication and the other five laws. This amounts to the declaration that plenty of algebraic theorems can be proved without recourse to Law III. (ii), thereby opening for algebra a far larger field of application, including, for instance, vectors, matrices, q-numbers for its more obvious annexations.

Algebra in fact has been undergoing an examination closely akin to that of geometry. Just as many geometrical facts may be proved on a basis which excludes the famous parallel postulate of Euclid, so also in algebra if we exclude the commutative law of multiplication. Algebra, so restricted, has a special name: it is called *Linear Associative Algebra*.

It would, of course, be possible to drop any one or more of these six laws and to follow up the consequences. To this there would be no logical objection: and it might even be entertaining. We might try to construct an algebra for which the law $a+b=b+a$ is not true: or for which *both* commutative laws fail. There would, however, be ample objection to this, for the simple reason that such a course would have no immediate application in geometry or physics, and would deprive algebra itself of the bulk of its hard-won results.

* A lecture delivered at the Leeds Meeting of the British Association, September, 1927.

A set of numbers arranged in a rectangular array

$$A=\begin{bmatrix}2 & 3 & 5 & 1\\ 4 & 6 & 2 & 3\\ 0 & 1 & 2 & 4\end{bmatrix}$$

is called a matrix. This example is a four by three matrix, and the twelve entries in the array are called the elements. Such elements may be real or complex numbers, which in turn may be treated as constants or variables. Simple examples of matrices are provided by sets of coordinates in geometry and sets of components of vectors in physics: they illustrate a matrix with a single row. Furthermore, as

$$[2,\ 4,\ 3],\quad [4,\ 2,\ 3]$$

would denote two distinct points, referred to a frame of three Cartesian axes *Oxyz*, so we agree to consider them as two distinct matrices.

It was Cayley who first saw the value of treating a block of several such rows as one thing, so extending the underlying idea of analytical geometry and the method of vectors. He found that such blocks could be treated as whole quantities which seldom need be decomposed into their elements, and in this way he was led to develop their properties on the basis of the first five laws of algebra, excluding the sixth. Matrices, in fact, can be denoted by letters A, B, C, which undergo two fundamental operations $+$ and $\times$, and satisfy the laws:

$$\begin{aligned}A+(B+C)&=(A+B)+C,\\ (A\times B)\times C&=A\times(B\times C),\\ A\times(B+C)&=(A\times B)+(A\times C),\\ (A+B)\times C&=(A\times C)+(B\times C),\\ A+B&=B+A,\end{aligned}$$

but not necessarily the sixth law, $AB=A\times B=BA$. Particular matrices may satisfy this, in which case they are said to commute with one another.

Briefly these laws are as follows. First the matrices can be made square by affixing, where necessary, rows or columns of zeros. For example we lay down the following equalities:

$$\begin{bmatrix}2 & 3 & 5 & 1\\ 4 & 6 & 2 & 3\\ 0 & 1 & 2 & 4\end{bmatrix}\equiv\begin{bmatrix}2 & 3 & 5 & 1\\ 4 & 6 & 2 & 3\\ 0 & 1 & 2 & 4\\ 0 & 0 & 0 & 0\end{bmatrix}$$

and

$$\begin{bmatrix}2 & 3\\ 4 & 6\\ 0 & 1\end{bmatrix}\equiv\begin{bmatrix}2 & 3 & 0\\ 4 & 6 & 0\\ 0 & 1 & 0\end{bmatrix},$$

extending when necessary a rectangular array to the right or downwards to make it square. Next we define addition of two squares of the same size, by adding corresponding elements. Thus, if

$$A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix},\qquad B=\begin{bmatrix}3 & 2\\ 1 & 0\end{bmatrix},$$

then

$$A+B=\begin{bmatrix}4 & 4\\ 4 & 4\end{bmatrix};$$

and it will be noted that this is a matrix of the same *order* as either A or B, two elements to a row or column.

Let a_{ij} denote the element of A in the i^{th} row and the j^{th} column, with a similar notation for B. Then we write for short:

$$A=[a_{ij}],\quad B=[b_{ij}],\quad A+B=[a_{ij}+b_{ij}].$$

Again, multiplication is defined by the following rule. From two square matrices $[a_{ij}]$ and $[b_{ij}]$, each with n rows and columns, we form a new square $[c_{ij}]$ of the same order n by taking

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

This is said to be the product $A \times B$ or briefly AB. So we write

$$C = AB.$$

This rule also applies to rectangular matrices, provided the lengths of the row of A and the column of B agree.

This rule applied to the above numerical examples would give

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3+2, & 2+0 \\ 9+4, & 6+0 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 13 & 6 \end{bmatrix} = C,$$

which is not the same as the product of B and A in the reverse order, since

$$BA = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3+6, & 6+8 \\ 1+0, & 2+0 \end{bmatrix} = \begin{bmatrix} 9, & 14 \\ 1, & 2 \end{bmatrix} = D.$$

Two matrices, C, D, are equal only when $c_{ij} = d_{ij}$ for each corresponding element, and this is certainly not the case here with AB and BA.

Subtraction is effected by defining $A - B$ as $[a_{ij} - b_{ij}]$. Any combination of addition, subtraction and multiplication of matrices of order n is always a matrix of order n. For instance:

$$AB - BA = C - D = \begin{bmatrix} 5 & 2 \\ 13 & 6 \end{bmatrix} - \begin{bmatrix} 9 & 14 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -4 & -12 \\ 12 & 4 \end{bmatrix}.$$

A simple theorem which follows from this is that the sum of the leading diagonal terms in the matrix $AB - BA$ is identically zero. The third order case makes this clear, for if

$$P = \begin{bmatrix} l & m & n \\ l' & m' & n' \\ l'' & m'' & n'' \end{bmatrix}, \qquad Q = \begin{bmatrix} \lambda & \lambda' & \lambda'' \\ \mu & \mu' & \mu'' \\ \nu & \nu' & \nu'' \end{bmatrix},$$

the leading diagonal terms of PQ are by definition

$$l\lambda + m\mu + n\nu, \quad l'\lambda' + m'\mu' + n'\nu', \quad l''\lambda'' + m''\mu'' + n''\nu'',$$

and of QP are

$$l\lambda + l'\lambda' + l''\lambda'', \quad m\mu + m'\mu' + m''\mu'', \quad n\nu + n'\nu' + n''\nu''.$$

Hence the sum of the leading diagonal terms of $PQ - QP$, which is that of PQ minus that of QP, is zero.

The rule of multiplication which at first sight seems rather artificial, occurs naturally enough when we consider the matrix as a linear transformation. Suppose, in fact,

$$\begin{aligned} x_1 &= ly_1 + my_2, \\ x_2 &= l'y_1 + m'y_2, \end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

and

$$\begin{aligned} y_1 &= \lambda z_1 + \lambda' z_2, \\ y_2 &= \mu z_1 + \mu' z_2. \end{aligned}$$

Such equations are said to transform x_1, x_2 to y_1, y_2, and again y_1, y_2 to z_1, z_2. If we adopt the matrix notation, and write

$$X = \begin{bmatrix} x_1, & 0 \\ x_2, & 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix},$$

we can regard X, Y, Z as points each with two coordinates. Further, if we form matrices from the coefficients,

$$L=\begin{bmatrix} l & m \\ l' & m' \end{bmatrix}, \qquad \Lambda=\begin{bmatrix} \lambda & \lambda' \\ \mu & \mu' \end{bmatrix},$$

our equations of transformation can be written compactly as

$$X=LY, \quad Y=\Lambda Z.$$

For the product matrix given by LY is precisely the same as X. This immediately gives us a physical interpretation of a matrix L. We first think of a point X with coordinates (x_1, x_2) referred to fixed Cartesian axes, and of another point Y with coordinates (y_1, y_2) referred to the same axes. Now if a number of such points X are connected with corresponding Y's by this set of equations (1), a definite geometrical law, such as a rotation or projection is at work. The matrix connotes this definite law. Similar remarks apply to n dimensions.

For example, the matrix

$$A=\begin{bmatrix} \cos\alpha, & +\sin\alpha \\ -\sin\alpha, & \cos\alpha \end{bmatrix}$$

would here denote a rotation of a vector OX through an angle α to the position OY. If α is zero, we write

$$I=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which is the unit matrix, answering to no change of position, the geometrical figure remaining unaltered. Again, if

$$K=\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix},$$

we find that the transformation $X=kY$ merely shifts the point X k units out along the line OY, so the effect of such a matrix K is to produce a similar figure similarly situated about the origin.

Matrices consisting of the same element k at each place in the leading diagonal, and of zeros elsewhere, are called scalar matrices. By the addition law we find

$$2I=I+I=\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

and in general

$$kI=\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix},$$

where k is real or complex.

The matrix K can thus be regarded as k times the unit matrix. Further, such matrices obey the commutative law as is easily verified, so they have the same six fundamental laws as ordinary numbers, also called scalar numbers.

In the same way we can derive a matrix kA from A, by multiplying each element of A by k. Thus, if

$$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \text{then } kA=\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}.$$

We can next evaluate A^2, A^3, etc., as matrices of order n, and even find a matrix equal to a polynomial such as

$$pA^3+qA^2+rA+sI,$$

where p, q, r, s are scalar numbers. Also the unit matrix commutes with other matrices, so that

$$AI=IA=A.$$

So it is not always necessary to write this unit factor I : and as in ordinary algebra it is frequently omitted. This leads to such expressions as $A+1$ for $A+I$, and $A-\lambda$ for $A-\lambda I$. So

$$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A+1=\begin{bmatrix} a+1, & b \\ c, & d+1 \end{bmatrix}, \quad A-\lambda=\begin{bmatrix} a-\lambda, & b \\ c, & d-\lambda \end{bmatrix}, \text{ etc.}$$

The array of elements of a square matrix A leads to a determinant $|A|$ called the determinant of the matrix. Thus

$$A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \qquad |A|=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

An immediate consequence of the product law $AB=C$ is that

$$|A|\times|B|=|C|.$$

If we denote the cofactor $b_2c_3-b_3c_2$ of a_1 in $|A|$ by a^1, and use upper suffixes for all such cofactors, we can form what is called the reciprocal matrix:

$$A^{-1}=\begin{bmatrix} a^1 & a^2 & a^3 \\ b^1 & b^2 & b^3 \\ c^1 & c^2 & c^3 \end{bmatrix}.$$

The name is justified, because the following relations are true, by use of a well-known theorem in determinants due to Cauchy:

$$AA^{-1}=A^{-1}A=I=1.$$

This leads to a division law $B\div A=B\times A^{-1}$, and therefore also to rational functions of a matrix.

It is well to observe that the fractional notation

$$\frac{A}{B}$$

is ambiguous, for it might mean $B^{-1}A$ or AB^{-1}, which we can call fore and after division by B. That there should be two kinds of division is really quite natural, because there are in fact two distinct elementary processes, *measuring* and *sharing*, which have place in elementary arithmetic. The distinction is shown clearly in the following:

$$£10:£5, \quad \tfrac{1}{5} \text{ of } £10,$$

and less clearly in $10:5$, $\frac{1}{5}$ of 10. It is entirely due to the commutative law that two such different processes should be merged into one arithmetical operation—division. Non-commutative algebra naturally disentangles them.

Sometimes, however, the fractional notation is legitimate, as in

$$\frac{1-A}{1+A} \text{ or even } \frac{f(A)}{\phi(A)},$$

where f and ϕ are polynomials with scalar coefficients. For we can prove

$$(1+A)^{-1}(1-A)=(1-A)(1+A)^{-1},$$
$$f(A)\times\{\phi(A)\}^{-1}=\{\phi(A)\}^{-1}\times f(A).$$

If the determinant $|A|$ vanishes, we cannot form the reciprocal matrix A^{-1}, any more than we can deal with the scalar number a^{-1} when a is zero. In this case the matrix A is said to be singular. Such singularity is graded into n *ranks*, denoted by 0, 1, 2, ... $n-1$, while n denotes the rank of the non-singular matrix. For a third order matrix these ranks are illustrated by

$$A=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B=\begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ 0 & 0 & 0 \end{bmatrix}, \quad C=\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 9 \end{bmatrix}, \quad D=\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \end{bmatrix}.$$

A has rank zero, when all its minor determinants are zero. B has rank one, when all its two-rowed minors are zero, but not all its elements. C has rank two, when its three-rowed determinant is zero, but not all its two-rowed minors. The non-singular third order matrix D has rank three.

We now have the following state of things for this non-commutative algebra of matrices: one type of addition and subtraction, as in ordinary algebra, but two types of multiplication and division which we can call fore and after

multiplication or division. Thus AX is the result of fore multiplication of X by A, and XA that of after multiplication.

There is also a new fundamental operation, called *transposition* and denoted by an accent. We write

$$A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \quad \text{and} \quad A'=\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

when rows and columns are interchanged. This operation satisfies the laws

$$A''=A,\ A'''=A',\ \text{etc.},$$
$$(A+B)'=A'+B',$$
$$(AB)'=B'A'.$$

The first is an obvious result, the next can easily be proved and calls for no remark. The last, however, is very typical of non-commutative algebra. Let us call it the reversal law. Such a law holds both for transposition and for reciprocation. In fact,

$$(AB)^{-1}=B^{-1}A^{-1},$$

since $$(AB)^{-1}(AB)=I,$$

or $$(AB)^{-1}(AB)B^{-1}=B^{-1},$$

or $$(AB)^{-1}A=B^{-1},$$

or $$(AB)^{-1}AA^{-1}=B^{-1}A^{-1}.$$

Finally, we have the other fundamental law, this time a commutative law,

$$(A')^{-1}=(A^{-1})'.$$

Simple examples of this calculus can now be given. First, if a matrix is unaltered by transposition it is symmetrical; and if every element changes sign thereby it is skew symmetrical. For a symmetrical matrix

$$A=[a_{ij}]=[a_{ji}]=A',$$

and for a skew symmetric matrix

$$S=[s_{ij}]=[-s_{ji}]=-[s_{ji}]=-S'.$$

The product BB' of any matrix and its transposed matrix is always symmetrical, for by the reversal law

$$(BB')'=B''B'=BB'.$$

A very important case of this is when B is a vector—a single row or column. The square of the resultant of a vector X can be written $X'X$, for if

$$X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad X'=[x_1,\ x_2,\ x_3],$$

then $$X'X=x_1^2+x_2^2+x_3^2,$$

a scalar quantity. Similarly for n elements.

A problem of great interest is to find the general orthogonal matrix with rational elements. Can we in fact find values of the nine (or n^2) elements

$$\begin{bmatrix} l & m & n \\ l' & m' & n' \\ l'' & m'' & n'' \end{bmatrix}$$

which answer to direction cosines of three (n) lines mutually at right angles? This is equivalent to finding a linear transformation from X to Y which leaves the square of the distance of a point from the origin unaltered, namely,

$$x_1^2+x_2^2+x_3^2=y_1^2+y_2^2+y_3^2.$$

or more compactly

$$X'X=Y'Y.$$

Cayley gave a pretty solution by using a subsidiary set of variables Z. If we translate the solution into matrix notation it provides an excellent illustration of the power of the method. We chose an arbitrary skew symmetric matrix S and the unit matrix I, so that $I'=I$, $S'=-S$. Next we take

$$\left.\begin{aligned} L&=I+S, & L'&=I-S \\ X&=LZ, & Y&=L'Z \end{aligned}\right\}.$$

Hence $L+L'=2$, and by the reversal law $X'=Z'L'$, $Y'=Z'L$.
Accordingly $X'X=Z'L'LZ$ and $Y'Y=Z'LL'Z$.
But $L'L=(2-L)L=2L-L^2=L(2-L)=LL'$.
Thus $X'X=Y'Y$, showing that the transformation from X to Y is orthogonal. But $Z=L^{-1}X=L'^{-1}Y$, so that $X=LL'^{-1}Y$. This further shows that the matrix M given by

$$M=LL'^{-1}=\frac{I+S}{I-S}$$

is orthogonal.

To calculate M, let the arbitrary skew matrix S with three rows and columns be

$$\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -c & 0 \end{bmatrix},$$

then

$$1+S=\begin{bmatrix} 1 & c & -b \\ c & 1 & a \\ b & -c & 1 \end{bmatrix}, \quad 1-S=\begin{bmatrix} 1 & -c & b \\ c & 1 & -a \\ -b & c & 1 \end{bmatrix},$$

from which $(1-S)^{-1}$ can be found, leading finally to

$$M=\begin{bmatrix} \dfrac{1+a^2-b^2-c^2}{1+a^2+b^2+c^2}, & \dfrac{2(ab+c)}{1+a^2+b^2+c^2}, & \dfrac{2(ac-b)}{1+a^2+b^2+c^2}, \\ \dfrac{2(ab-c)}{1+a^2+b^2+c^2}, & \dfrac{1-a^2+b^2-c^2}{1+a^2+b^2+c^2}, & \dfrac{2(bc+a)}{1+a^2+b^2+c^2}, \\ \dfrac{2(ac+b)}{1+a^2+b^2+c^2}, & \dfrac{2(bc-a)}{1+a^2+b^2+c^2}, & \dfrac{1-a^2-b^2+c^2}{1+a^2+b^2+c^2}. \end{bmatrix}$$

This is closely connected with results known to Rodrigue and to Euler, together with formulae of use in the theory of quaternions.

It may be asked, have we found the most general orthogonal matrix? To this the answer is no, because we have not taken into account a possible change from a right to a left-handed system of axes: nevertheless M contains the maximum number of arbitrary parameters a, b, c consistent with orthogonality. In general this number is $\frac{1}{2}n(n-1)$.

An important corollary follows, which concerns the general quadratic relation

$$\Sigma a_{ij}x_ix_j=\Sigma a_{ij}y_iy_j$$

analogous to the simpler case already taken. Can a linear transformation of variables

$$X'=[x_1, x_2, \ldots x_n], \quad Y'=[y_1, y_2, \ldots y_n]$$

from X to Y be found which leaves the above quadratic relation unaltered? The answer is quite simple, for we can write the coefficients a_{ij} of a general quadratic as a *symmetrical* matrix

$$Q=[a_{ij}]$$

and the quadratic itself as a matrix product

$$X'QX.$$

For instance,

$$[x_1, x_2, x_3]\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = ax_1^2 + bx_2^2 + cx_3^2 + 2fx_2x_3 + 2gx_3x_1 + 2hx_1x_2.$$

Our condition is now $X'QX = Y'QY$, and this is secured by taking as our linear transformation

$$X = \frac{1+SQ}{1-SQ}\,Y,$$

where S is an arbitrary skew symmetric matrix. For by transposition

$$X' = Y'\frac{1-QS}{1+QS},$$

since $Q' = Q$, $S' = -S$. Hence $X'QX = Y'QY$, provided

$$\frac{1-QS}{1+QS}\,Q\,\frac{1+SQ}{1-SQ} = Q.$$

Multiplying fore and aft by $1+QS$ and $1-SQ$ respectively, we have

$$(1-QS)Q(1+SQ) = (1+QS)Q(1-SQ)$$

which is true on expansion without the commutative law of multiplication.

An important point arises from the theory of factors of matrices: it does not follow that in every instance either of our two types of division is a unique operation. For since AB can vanish without either factor vanishing, we cannot infer from the supposition $AP = AQ$ that $P = Q$. In fact,

$$AP - AQ = 0,$$

i.e.

$$A(P-Q) = 0,$$

does not lead to $P - Q = 0$. Then, if $D = AP = AQ$, we can consider both P and Q as results of dividing D by A, which means that division is not a unique operation. This leads to the question what are the conditions for division to be unique? This was answered by Frobenius * forty years ago, who proved the theorem, that division was unique for linear associative algebra only in the following cases: the algebra of ordinary real number, that of complex number, and that of quaternions. Of these the last alone is non-commutative.

Now the usual proof of this theorem applies to finite matrices: it may or may not hold for infinite matrices. The answer to this again affects the theory of another type of non-commutative number called q-number, for it has been assumed as a postulate of q-numbers that division is unique. So unless the theorem provides a loophole when the order of the matrix is infinite, the postulate is untenable.

At this stage it is perhaps well to remark that certain relations are structurally impossible for matrices. For instance, it would be impossible to deal with the equation

$$L + L' = A$$

where A is a given unsymmetrical matrix, and L has to be found. For $L + L'$ is necessarily symmetrical.

More interesting is the fact that such an equation as

$$PQ - QP = k$$

* Crelle, Bd. 84. A proof is given by L. E. Dickson in the Cambridge University Tract *Linear Algebras* (1914), p. 10.

cannot hold, if k is a non-zero scalar quantity. For if P is a square matrix of order n, so also must Q be, in order to multiply P by Q and Q by P. Then the sum of the diagonal terms is zero on the left, but is nk on the right. Whence

$$0 = nk,$$

so that k is zero, which contradicts the hypothesis.

The same applies if P is an $n \times m$ matrix, which requires Q to be an $m \times n$ matrix.

It happens that a recent physical investigation has brought to light an equation of this type, involving quantities to which allusion has already been made, called q-numbers, obeying the first five laws of algebra but not the sixth. So if p, q are such q-numbers, we do not assume

$$pq - qp = 0.$$

Now an equation of great importance, which involves two of these q-numbers together with ordinary scalar numbers h, π, i, is

$$pq - qp = \frac{h}{2\pi i}.$$

Also it has been suggested that such q-numbers can be regarded as matrices each with an infinite number of rows and columns. This at once raises the interesting question: can the relation

$$PQ - QP = k$$

hold for infinite matrices, in spite of its failure for finite orders? This takes us at once out of the realms of algebra into those of analysis.

If

$$P = \begin{bmatrix} p_{11} & p_{12} \cdots \\ p_{21} & p_{22} \cdots \\ \cdots\cdots\cdots \end{bmatrix}, \quad Q = \begin{bmatrix} q_{11} & q_{21} \cdots \\ q_{12} & q_{22} \cdots \\ \cdots\cdots\cdots \end{bmatrix},$$

let

$$R = \begin{bmatrix} p_{11}q_{11}, & p_{12}q_{12}, \cdots \\ p_{21}q_{21}, & p_{22}q_{22}, \cdots \\ \cdots\cdots\cdots\cdots\cdots \end{bmatrix},$$

which is a matrix formed by multiplying corresponding elements of P and Q together. Further let the sum of the elements in the ith row of R be R_i, and in the jth column of R be C_j; then the equation $PQ - QP = k$ requires, among other conditions,

$$k = R_1 - C_1 = R_2 - C_2 = R_3 - C_3 = \ldots,$$

so that $R_1 + R_2 + R_3 + \ldots - C_1 - C_2 - C_3 - \ldots$ must be infinite. Accordingly the double series whose terms are those of R summed by rows must differ infinitely from the same series summed by columns!

That this unusual requirement—which outrages all our theories of book-keeping and accountancy—is possible, shall now be shown.

Let

$$P = \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & 0 \ldots \\ -\sqrt{1} & 0 & \sqrt{2} & 0 & 0 \ldots \\ 0 & -\sqrt{2} & 0 & \sqrt{3} & 0 \ldots \\ 0 & 0 & -\sqrt{3} & 0 & \sqrt{4} \ldots \\ \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & 0 \ldots \\ \sqrt{1} & 0 & \sqrt{2} & 0 & 0 \ldots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 \ldots \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} \ldots \\ \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \end{bmatrix},$$

$$PQ = \begin{bmatrix} 1 & 0 & \sqrt{2} & 0 & 0 \ldots \\ 0 & 1 & 0 & \sqrt{6} & 0 \ldots \\ -\sqrt{2} & 0 & 1 & 0 & \sqrt{12} \ldots \\ 0 & -\sqrt{6} & 0 & 1 & 0 \ldots \\ \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \end{bmatrix}, \quad QP = \begin{bmatrix} -1 & 0 & \sqrt{2} & 0 & 0 \ldots \\ 0 & -1 & 0 & \sqrt{6} & 0 \ldots \\ -\sqrt{2} & 0 & -1 & 0 & \sqrt{12} \ldots \\ 0 & -\sqrt{6} & 0 & -1 & 0 \ldots \\ \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \end{bmatrix},$$

so that $PQ-QP=2$. In this case

$$R=[p_{ij}q_{ij}]=\begin{bmatrix} 0 & 1 & 0 & 0\dots \\ -1 & 0 & 2 & 0\dots \\ 0 & -2 & 0 & 3\dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}=[r_{ij}].$$

The double series $\Sigma\Sigma r_{ij}$ summed by rows is

$$1+1+1+\dots,$$

and yet summed by columns it is

$$-1-1-1-\dots.*$$

On the other hand, if p, q are not matrices there is no logical difficulty in assuming that a relation which makes $pq-qp$ scalar is possible. Indeed, the equation

$$xa-ax=1$$

is compatible with the first five laws of algebra apart from the sixth, because actual entities exist which behave in this way. For instance, if a is an ordinary number, and y is a function of a and x denotes the operator $\frac{d}{da}$, then the expression

$$(xa-ax)y=y$$

is only another way of writing the identity

$$\frac{d}{da}ay-a\frac{d}{da}y=y.$$

As this is true for all differentiable values of y, we can write symbolically

$$\frac{d}{da}a-a\frac{d}{da}=1.$$

Also such entities as $\frac{d}{da}$, a obey the other laws of algebra but not the commutative law.

It is interesting to manipulate such equations, and to deduce results very like ordinary differentiation. From $xa-ax=1$ we can deduce

$$\begin{aligned} x^2a-ax^2&=2x,\\ x^na-ax^n&=nx^{n-1},\\ x^na^n&=xa(xa+1)(xa+2)\dots(xa+n-1),\\ a^nx^n&=ax(ax-1)(ax-2)\dots(ax-n+1); \end{aligned}$$

and indeed Dr. Dirac has utilized this idea to define the differential coefficient of a q-number z with regard to x. First it is supposed that q numbers, which include as particular cases scalar numbers, exist which commute with x. Then if all values of b which satisfy $xb=bx$ also satisfy $zb=bz$, z is said to be a function of x. Further, if $xa-ax=1$ leads to $za-az=z_1$, this last is defined as the differential coefficient of z with regard to x. It gives a definite value to z_1, the same for each value of a, for if possible let

$$\begin{aligned} za-az&=z_1, \qquad xa-ax=1,\\ zb-bz&=z_2, \qquad xb-bx=1, \end{aligned}$$

then $$z(a-b)-(a-b)z=z_1-z_2, \quad x(a-b)-(a-b)x=0.$$

Accordingly $a-b$ commutes with x. But because z is a function of x, then $a-b$ will also commute with z; whence

$$z(a-b)-(a-b)z=0,$$

* I understand, from Dr. Heisenberg, that the conditions of convergency or divergency inherent in such matrices have quite recently been studied by Prof. Hilbert.

so that $z_1 = z_2$. Furthermore this derivate z_1 of z satisfies the usual addition and product laws of ordinary differentiation.

Lastly it may be remarked that a square matrix $X = [x_{ij}]$ whose n^2 elements are regarded as independent variables, leads to an interesting differential operator matrix which we may write

$$\Omega = \left[\frac{\partial}{\partial x_{ji}}\right] = \begin{bmatrix} \frac{\partial}{\partial x_{11}} & \frac{\partial}{\partial x_{21}} & \cdots & \frac{\partial}{\partial x_{n1}} \\ \frac{\partial}{\partial x_{12}} & \cdots\cdots & & \frac{\partial}{\partial x_{n2}} \\ \cdots\cdots & & & \\ \frac{\partial}{\partial x_{n1}} & \cdots\cdots & & \frac{\partial}{\partial x_{nn}} \end{bmatrix}.$$

By transposition we have an analogous operator $\Omega' = \left[\frac{\partial}{\partial x_{ij}}\right]$. If A is a square matrix whose n^2 elements are constants, we obtain the result

$$\Omega A X' = A'$$

or

$$\left[\frac{\partial}{\partial x_{ij}}\right][a_{ji}][x_{ij}] = [a_{ij}],$$

which shows that the operation of transposing A to A' or A' to A can be looked on as a matrix differential operation.

The operator Ω is a generalization of ordinary differentiation, for if the matrix has only one element, n being unity, then $X' = X = x$ say, and $\Omega = \frac{d}{dx} = \Omega'$; and transposition loses its peculiar significance.

In like manner we obtain

$$\Omega X'^p = X'^{p-1} + X X'^{p-2} + \ldots + X^{p-1}$$

as an interesting generalization of $\frac{d}{dz} x^p = p x^{p-1}$, and the very curious formula when Ω operates on an integral power, positive or negative, of X,

$$\Omega X^p = \sum_{i=1}^{n} \frac{X^p - \lambda_i{}^p}{X - \lambda_i},$$

where $\lambda_1, \lambda_2, \ldots \lambda_n$ are the latent roots of the matrix X: by which is meant the n roots of the equation $|X - \lambda I| = 0$.

It can be shown that such an operator is a very effective tool in discussing the theory of a function $f(X)$ involving one matrix X along with scalar quantities.

H. W. TURNBULL.

GLEANINGS FAR AND NEAR.

479. I am sometimes forced to make types . . . of which I here send you a specimen ($a \pm b \pm c$). It is called *plus-minus* $\pm$. I printed my first Tracts at Cambridge when Archdeacon (not Bentham) was their Printer. I was very sick of it; the University meanly provided with mathematical types, insomuch that they used daggers turned sideways for *plus's*.—The Rev. William Ledham Nichols, *Literary Anecdotes*, 1812-1858, vol. viii. p. 414.

480. One cannot wonder at the bravery of the British officers at Waterloo if they had mastered [the *Course in Mathematics* by Chas. Hutton, LL.D., F.R.S.].—J. L. Coolidge, *The American Mathematical Monthly*, Feb. 1926, p. 70.

NOETHER'S CANONICAL CURVES.

By Prof. W. P. Milne, D.Sc.

(*Short report of Lecture delivered before the British Association*, 1927.)

Noether has shown that the general plane curve of degree n and genus p can be represented point by point on a canonical curve of degree $2(p-1)$ in space of $(p-1)$ dimensions. It is, therefore, of the greatest importance in curve theory to obtain the properties of these canonical curves, and in particular those properties most nearly allied to the properties of the periods of the corresponding Abelian Functions.

The canonical curve for $p=3$ is the plane quartic curve. Two 4-tangent conics are said to belong to the same system when a conic passes through their eight points of contact with the quartic. The polar conics with respect to a cubic curve of points on a fixed conic constitute the complete family of 4-tangent conics of the same system of a quartic curve. The polar conics of the six points in which the fixed conic meets the Hessian of the cubic curve constitute the six pairs of bitangents that form the degenerate members of the system.

The canonical curve for $p=4$ is the quadri-cubic curve, *i.e.* the curve of intersection of a cubic surface with a quadric. Two 6-tangent quadrics are said to belong to the same system when a quadric (not containing the quadri-cubic curve) passes through their twelve points of contact with the quadri-cubic curve. The points of contact of the complete family of 6-tangent quadrics of the same system are defined by the points of contact of the polo-cones with respect to a cubic surface of a system of straight lines lying in a fixed plane (the polo-cone of a given line being defined as the locus of points whose polar quadrics with respect to the cubic surface touch the given line). The quadri-cubic curve in this case is the curve of intersection of a given quadric with the polo-cubic surface of the above cubic surface and the fixed plane (*i.e.* the locus of points whose polar quadrics touch the given plane). The locus of points in the above fixed plane, whose polar planes with respect to the cubic surface touch the given quadric is a quartic curve, the polo-cones of whose bitangents have double contact with the given quadric and therefore intersect the quadric in a quadri-quadric curve which degenerates into two conics, the planes of which constitute the 28 pairs of tritangent planes of the same system of the quadri-cubic canonical curve.

A discussion of the particular case for $p=6$, typified by the plane quintic curve, can be found in my paper on "The 5-tangent conics of the plane quintic curve" (*Journal of the London Math. Soc.* vol. 2, part 2). Further references to the subject can be found in a paper by Roth (*Monatsh. Math. Phys.* 22 (1911), 64-88), and Guyaelf (*Proc. London Math. Soc.* Ser. 2, vol. 21, part 5, and *Proc. London Math. Soc.* Ser. 2, vol. 25, part 3). W. P. Milne.

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481. *Marmaduke Multiply's Merry Method of making Minor Mathematicians*; or the Multiplication Table illustrated by sixty-nine appropriate engravings, by I. F. C. 16mo. London, printed for J. Harris. No date, but probably about 1820. It is difficult to exaggerate the interest and charm of the 69, well calculated to drive home wholesome mathematical and other truths in the mind of the dullest (*Athenaeum, circ.* 1840).

482. An unconscious example of the ludicrous extent to which a clever man will remove trifles in the path: The pudding is to be eaten with a knife and fork, beginning at the circumference of the slice, and approaching regularly towards the center, each piece of pudding being taken up by the fork, and dipped into the butter, or dipped into it in part only, as is generally the case, before it is carried to the mouth.—Rumford's *Essays*, i. p. 267, 1802.

REVIEWS.

The Theory of Functions of a Real Variable and the Theory of Fourier's Series. Vol. I. 3rd Ed. By E. W. HOBSON. Pp. xv+736. 45s. net. 1927. (Cam. Univ. Press.)

Fourteen years passed between the first and second editions of this book; the second and third are separated by only six. As the third edition of Volume I. appears so quickly after the second edition of Volume II., it seems that Prof. Hobson must be continuously engaged in revising or rewriting some part or other of his great treatise. If he is ever disposed to lament the burden of this work, we hope that he will find consolation in the thought that the quick sale of the second edition shows the high esteem in which the book is held.

This edition is sixty pages longer than the second. The chief single addition is a rewritten account of the Riemann-Stieltjes integral—the extension of the ordinary Riemann integral in which the variable of integration is replaced by a function of that variable. Prof. Hobson shows that the definition of the integral is a matter of some nicety and that the definitions as a limit and as a common bound which are equivalent for the Riemann integral are no longer equivalent for the more general concept.

Another new section which we welcome is that on Vitali's theorem on the covering of a set of points by a set of intervals of nearly the same measure. The integral of a function of two variables receives an extended treatment.

Our only regret is that the printer did not at times protest against the complexity of the notation and persuade the author to reduce such a welter of brackets and suffixes as appears on page 615. J. C. BURKILL.

Harrap's Introductory Algebra. By W. FARQUHARSON and H. W. CARTER. New and enlarged edition. Pp. 180. 2s. 6d. 1927. (G. G. Harrap & Co.)

The first edition, published in 1917, was reviewed in the *Gazette*, vol. ix. p. 183. The usual ground is covered up to solution of quadratics by formula, with short sections on the use of squared paper and the graphical solution of two linear equations, or of one quadratic. The chief feature remains the analysis of the elementary principles of algebra by an abundance of short questions. As was remarked in the review of the first edition, there is a danger that the "human boy" may take what seems the line of least resistance and seize on the summary rules printed in heavy type at the end of each section of analysis. As some of these rules are inevitably cumbrous, he will probably also misinterpret them. If this can be sufficiently discouraged by the teacher, the book gives good value; print and paper are good. There are 31 pages of revision papers, but no answers.

An Introduction to Practical Mathematics. By F. M. SAXELBY. New edition. Pp. vi+232. 4s. 1927. (Longmans.)

The first edition of this book, published in 1908, was noticed in the *Gazette*, vol. iv. p. 395. Both it and the author's larger "Course in Practical Mathematics" are well known to many readers of the *Gazette*. The last chapter of the previous edition, on rate of increase, has in this edition been replaced by one on geometry. It appears from the preface that this has been done in deference to the Association's Report on the Teaching of Mathematics to Evening Technical Students, 1926. It is for such students that the book is designed. In the new chapter, the fundamental facts concerning parallels, congruence, etc., are discussed by appeal to experience, and there follow deductive proofs of a few propositions of elementary geometry selected for their practical value. Their use is illustrated by exercises, which are well adapted to the interests of evening students, as are those in the previous chapters. The tables bound in at the end are criminally fatiguing to the eyes. Indeed the type used for the examples throughout the book, though clear, is definitely too small. Perhaps some of us will live to see a Socialist government legislate on this matter.

Elementary Mathematics. By A. M. Bozman, A. E. Denmark and E. M. Trickey. Pp. x+140. 2s. 6d. 1927. (University of London Press.)

This "combined course in arithmetic, algebra, and geometry for the junior forms of secondary schools" is compiled by mistresses of the County Secondary School at Streatham. It owes its conception, to quote the preface, "to the complete lack of suitable text-books for use in the junior forms of a school run on a modified Dalton plan." Many groups of examples are accordingly classified into lower, middle and higher assignments. The geometry is deductive, based on assumptions concerning congruence, parallels and supplementary angles, stated explicitly and supposed to be familiar from previous practical work. So far as a reviewer inexperienced in the Dalton plan can judge, the book should prove attractive to pupils and serve its purpose well. Answers are given.

Elementary Mathematics. Part III. By E. Sankey and A. Royds. Pp. viii+152. 2s. 1927. (G. Bell & Sons.)

Parts I. and II. were noticed in the *Gazette*, vol. xiii. pp. 179 and 434. The present part is intended for pupils of 13 to 15 years of age in central and senior elementary schools, and deals chiefly with contracted calculation, interest, factors, quadratics, simple mensuration, deductive geometry of parallels and triangles, and the beginnings of trigonometry, followed by 9 pages of problems and tables of logs, etc. Exercises occupy about half the space. Print and paper are satisfactory. There are no answers. The treatment throughout is conservative, proofs by superposition and Playfair's axiom being retained. In spite of preliminary advice to pupils to give up answers in a reasonable form, results of specimen examples on pp. 127 and 128 are given to 5 and 6 figures deduced from 4-figure tables.

School Trigonometry. Part II. By B. A. Howard. Pp. vi+90. 1s. 9d. 1926. (Ginn & Co.)

This is a sequel to the author's "First Ideas of Trignometry" which was favourably reviewed in the *Gazette*, vol. xii. p. 295. The present part deals with angles of any magnitude, circular measure, trigonometrical graphs and their use in solving otherwise intractable equations (a specially satisfactory piece of work), the use of the tan $\frac{A}{2}$ and tan $\frac{B-C}{2}$ formulae in the solution of triangles, and the more elementary identities, equations and properties of the triangle. The gap between the host of books on numerical trigonometry on the one hand and the more formal textbooks on analytical trigonometry on the other is very competently and usefully filled by this well-printed and inexpensive little book. Perhaps it is ungrateful to complain that the cover is not sufficiently durable and that the tables at the end are so crowded as to be worse than useless. We would also ask that in the next edition the examples on circular measure may be extended to cover the solution of triangles in which one angle is small; surely this is the chief practical application of the theory. Answers are issued separately to teachers only, price 6d.

Tables de Calcul. By Tsuneta Yano. Pp. iii+1816. 16s. post free. 1927. (The Actuary, The First Mutual Life Insurance Co., The First Mutual Building, Kyobashi, Tokio, Japan.)

An explanatory preface in Japanese is followed by the exact products of natural numbers to 999 times 999, each multiplier governing a two-page opening. A short table gives seven-figure logarithms of the numbers 1 to 1509. Finally there are 12 pages, unintelligible without a knowledge of Japanese, which seem to deal with actuarial calculations.

The net result for English users is a one volume equivalent of Crelle's *Rechentafeln*, with pages measuring $4\frac{1}{4}''$ by $7\frac{1}{2}''$. The paper is thin, but seems reasonably tough: whether the shorter life of the volume compared with that of the current edition of Crelle would more than compensate for the cheaper price could only be determined by use. The binding is half leather and half

buckram and the book lies open easily. Columns and lines are spaced in modern fashion, and the printing is reasonably clear; but the digits are of even height without heads and tails. If this is the only product table of portable size in print, it seems to meet a want. C. W. A.

Initiation aux Méthodes Vectorielles. By G. BOULIGAND and G. RABATÉ. Pp. viii + 216. 20 fr. + 40%. 1926. (Vuibert.)

The temptation to teach incidentally some subject other than their own is one which writers on vector analysis seem to find irresistible. As a rule, their zeal to make clear the value of their methods leads them to expound the elements of almost all the branches of applied mathematics and mathematical physics. Prof. Bouligand and M. Rabaté confine themselves to pure mathematics, but a quarter of their book is an introduction to the geometrical study of differential equations, and to this section, in spite of occasional use of vectorial terms, the vectorial setting is quite irrelevant.

The first four chapters of the book contain the elementary algebra of vectors and the theory of rotors. Chapter five proceeds from the derivative of a vector to Frenet's formulae, and is followed by a short chapter on the gradient of a scalar function of position. Then comes a long chapter divided into two sections. The first section is on integration, and ends with the theorems of Stokes and Green; the exposition is clear, but an enthusiast will feel that Cartesian coordinates play an unnecessarily prominent part, and will point his criticism by asking whether the reader ought to feel confident that the rotation and the divergence of a vector field are independent of the choice of axes. The second section of this chapter is the elementary treatment of differential equations to which reference has already been made; in itself it is an admirable piece of work, and in fact it is the best introduction to the whole subject that I happen to have come across, wherefore it is the more to be regretted that the authors have buried it where nobody will dream of looking for it. The last chapter is on the theory of surfaces. Spherical representation is used to set up a correlation between vectors tangential to the surface and vectors tangential to a unit sphere. At corresponding points of the surface and the sphere, this correlation determines a linear transformation from points in the one tangent plane to points in the other, and curvature is discussed by means of this transformation. This line of development involves a parenthetical treatment of the linear transformation, rightly limited to two dimensions, for after all this is an elementary book, and a simple case which is wanted for immediate application is a stimulating introduction to any theory. A collection of exercises forms a useful appendix.

The authors share none of my own fads, and in the little matter of the sign attached to the torsion of a curve I think the arguments are all against the authorities whom they follow. In a number of places, as for example with regard to the theorem that a twisted curve is effectively determined by the curvature and the torsion, they present a result quite properly as intuitive and add that it can be established rigorously; the students for whom this book is meant will wonder in what a real proof consists, and references might have served the excellent purpose of encouraging them to satisfy their curiosity. But the book is to be warmly recommended. The study of vectorial methods is made attractive and the wide range of their application is indicated clearly. I do not think it is an exaggeration to say that a better introduction to the subject does not exist. E. H. N.

The Yearbook of the Universities of the Empire. 1927. Edited by W. H. DAWSON. Pp. xii + 858. 7s. 6d. net. (Bell & Son.)

The Yearbook has been with us since 1914. It now forms a thoroughly reliable and convenient digest of the official Calendars of seventy-one Universities. It is a directory of the officers and staff of each University, with all the information as to equipment, degrees, fees, scholarships, accommodation, etc., which the enquirer may need with respect to any of these institutions. Among the novelties in the current edition is an appendix giving lists of titles of theses for research degrees. The Yearbook is published under the aegis of the Universities' Bureau of the Empire.

Sir Isaac Newton. A Brief Account of his Life and Work. By S. BRODETSKY. Pp. xii+161. 5s. net. 1927. (Methuen.)

It is remarkable that but one full-dress biography of Newton exists in the language. Short lives there are in plenty. We have them in collections such as the *Biographia Britannica*, Benjamin Martin's *Biographia Philosophica*, the *Penny Cyclopaedia*, and the like, ending with the *D.N.B.* Lord Brougham (1829) wrote one for the Library of Useful Knowledge, but it was little more than a paraphrase of that by Biot in the *Biographie Universelle*. Brewster's small *Life* (1831) was in existence nearly thirty years before the *Memoirs* made their appearance. P. E. B. Jourdain did yeoman's service by collecting all the essays and reviews dealing with Newton that came between 1846 and 1858 from the learned and indefatigable pen of De Morgan, and by publishing them in one volume in 1914. The uninstructed enthusiasm of the hero-worshippers was roused to frenzy by De Morgan. who was roundly denounced for his unpatriotic action in drawing the frailties of the popular idol "from their dread abode." But his opinions remained without serious challenge, and the opportunity was lost for that second full-dress biography, the absence of which we have still to deplore.

To Professor Brodetsky we are all grateful for the pleasant little volume of some 160 pages in which he has given an account of the life and work of the immortal whose bi-centenary has just been celebrated. He has set out to give a picture of the man, together with such a description of the work accomplished by his consummate genius as will be within the comprehension of the ordinary reader or the better boys in our schools. He has achieved a signal success. It is fortunate, indeed, that to write such a volume should have fallen to a mathematician who has the pen of a ready writer, the dramatic skill to present his readers with something that is more than a lay figure, and a welcome power of lucid exposition. His style is happy, and he preserves the appropriate mean between technical accuracy of statement and the simple, unaffected narrative. We learn something of the wonderful age in which imagination was replaced by experiment. We become aware of the nature of the difficulties that Newton had to face, and of the relentless persistence with which he overcame them. We appreciate more than ever the saying of the leader of the "Trimmers": "A difficulty raiseth the Spirits of a great man: he hath a mind to wrestle with it and give it a fall. A man's mind must be very low if his difficulty doth not make part of his pleasure." We understand Newton's resentment at being forced into a contest, and why he said that he was parting with his peace for a shadow.

Here we have fulness within a little compass. With Einstein our author pays his tribute of "reverent admiration to this shining spirit, who pointed out as none before or after him did, the path of Western thought and research and practical construction." To his life: "rarely, if ever have so many noble qualities been united in one man." To his work: "The impression that Newton made upon British scientific life, and on the scientific life of humanity as a whole, has remained one of the indelible imprints on the sands of time."

It is a notable achievement that after reading so slender a volume such encomiums should seem but natural and without exaggeration.

Prof. Brodetsky now and again indulges in timely and characteristic reflections. A couple of instances will suffice.

"Now the Greeks knew practically nothing about the science of dynamics (the other contemporary nations knew next to nothing of anything)."

"The school boy (or girl) of to-day is taught so well that one often wonders if he can learn at all. Good teaching does not consist in reducing all the hard nuts to an amorphous powder, and all the hard crusts to soft pap, which can be swallowed without effort, and which escape assimilation into the system owing to the lack of the saliva of mental effort. The learner must learn and not merely be taught; the effort must at least be his as much as the teacher's. When mathematical teaching is a careful cataloguing of all possible questions that might be invented by evil-minded examiners, and literary study has become a careful cataloguing of insignificant events in the authors' lives, coupled with pathological investigations into peculiarities of accidence, syntax, and misprints of successive editions, who will wonder that grown-up English-

men do not rush to see Shakespeare's dramas, and do not feel the romance of the binomial theorem? Yet the binomial theorem is full of romance; it throbs with human interest. It is an adventure into the unknown."

Several slips need attention in a future edition. For instance, who but Puck himself could have shifted a word from line 3 to the space above on line 2 and left the amusing statement on p. 128? On p. 54, l. 4 up, a missing "not" would be useful. But those who are familiar with the vagaries of proof-reading are never surprised at such occurrences. An editor who ought to have known better felt that our Newton memorial volume was spoiled by leaving *Tibi* for *Sibi* on p. vii, l. 3 up.

Apropos of the remark that as the first edition of the *Principia* became scarce its price went up, we have the first-hand testimony of Sir William Browne, M.D., the founder of the Browne Medals at Cambridge. In a speech advocating a knowledge of mathematics as one of the qualifications for occupants of the Chair of the Royal Society (*v.* Mayor's *Cambridge under Queen Anne*, 1911, p. 547) he states that the book, "originally of but ten or twelve shillings price, had risen so high above par, that I gave no less than two guineas for one, which was then esteemed a very cheap purchase."

The idealised bust which is the frontispiece of this volume is by Roubiliac, who took the cast of Newton's face, now in the Cambridge University Library. Many have been puzzled by Cipriani's stained-glass window set up (1774-5) at the south end of the Trinity Library. In 1813 Maria Edgeworth described it as follows: "It represents George III seated on his throne, and Bacon seated on the steps, writing. The King had been Henry VIII, but the Fellows made Cipriani cut off his head, and put that of George III in its place. The junction is still to be seen in the first design of the picture, covered with a pasted paper cravat." In a more sober statement, Dr. Sinker tells us that: "to bring in the most famous sons of Trinity we have here Newton presented to George III by a figure apparently representing Fame, while Bacon sits by as though recording the fact." The late Mr. P. E. B. Jourdain ingeniously suggested that in this anachronistic conjunction of figures we may have a reason for Rosenberger's description of Bacon as a friend of Newton. The window was paid for from a legacy left to the College by the Master who founded the Smith's prizes.

With regard to Newton's London homes, it is not unlikely that when made Warden of the Mint in 1697 he selected Jermyn Street because previously acquainted with its suitability for his requirements. For on 2nd Jan., 1689, Flamstead addressed a letter to: "Mr. Izaak Newton, at his house in German Street, near St. James's." He had been living at Chelsea for a year before he went to 35 St. Martin's Street. Pilgrims will look in vain for this little house, of which Macaulay writes: "It is still well known and will continue to be well known as long as our island retains any trace of civilisation; for it was the dwelling of Newton, and the square turret which distinguishes it from the surrounding buildings was Newton's Observatory." The house was also dear to lovers of literature. Dr. Burney lived at No. 35 in later years, and Fanny Burney wrote *Evelina* in the house "over-looking all London," which was surmounted by Wren's Observatory.

For raising Hugon's great telescope at Wanstead, Newton purchased part of the last London Maypole (? in the Strand or in Mayfair).

Readers may be interested in the following letter from Newton to his friend Dr. Law, for it reveals a certain geniality of temperament not usually associated with the great thinker. It was shown at a conversazione given to Helmholtz in University College, and is copied from *Nature*, 12th May, 1881, vol. xxiv. p. 39.

LONDON, *Dec.* 15, 1716.

DEAR DOCTOR: He that in ye mine of knowledge deepest diggeth, hath, like every other miner ye least breathing time, and must sometimes at least come to terr; alt for air.

In one of these respiratory intervals I now sit doune to write to you, my friend.

You ask me how, with so much study, I manage to retene my health. Ah, my dear doctor, you have a better opinion of your lazy friend than he hath

of himself. Morpheous is my best companion; without 8 or 9 hours of him yr. correspondent is not worth one scavenger's peruke. My practizes did at ye first hurt my stomach, but now I eat heartily enow as y' will see when I come down beside you.

I have been much amused by ye singular *φενομενα* resulting from bringing of a needle into contact with a piece of amber or resin fricated on silke clothe. Ye flame putteth me in mind of sheet lightning on a small—how very small—scale. But I shall in my epistles abjure Philosophy whereof when I come down to Sakly I'll give you enow. I began to scrawl at 5 minutes from 9 of ye clk, and have in writing consmd 10 mins.

My Lord Somerset is announced.

Farewell, Gd bless you and help yr sincere friend

ISAAC NEWTON.

To Dr. Law, Suffolk.

Apropos of Newton's handwriting, a reviewer of the Newton memorial volume in *The British Medical Journal* has stated that he does not think it generally known that Newton was myopic, "yet surely none but a short-sighted man could have written the minute script of his 'Scheme for prosecuting counterfeiters.' . . ." That is, of course, the judgment of an expert, writing of a man of at least fifty-four years of age, who, according to Conduitt, enjoyed such general health that in all his life he lost but one tooth, and had "a very lively and piercing eye."

We do not think that anyone but De Morgan has recorded an examination in which Newton was an unsuccessful candidate, and he does little more than note the fact. The following details about the successful competitor may interest the reader.

At the funeral of the Lord Protector, a little Westminster boy, stung beyond endurance at the sight of royal emblems upon the bier of the hated Cromwell, sprang forward, snatched away the little satin banner known as the Majesty scutcheon, darted back and escaped. Robert Uvedale was the name of the lad, and the trophy was long a much-prized treasure in his family. A plate of the scutcheon may be seen by the curious in *The Memoirs of the Verney Family*, 1894, vol. iii. p. 424.

In due course the boy entered at Trinity, Cambridge, and taking his degree in 1662 was elected to a Divinity Fellowship. Newton took his B.A. in 1665, and as the volume under review tells us "we know he was not elected Fellow at once." It was at this juncture that a Law Fellowship fell vacant. Uvedale and Newton were competitors: if there were others, there was no question as to their being in the running. Barrow himself was the examiner and found that Uvedale and Newton were equal in literary attainments. He therefore awarded the Fellowship to Uvedale as the senior. Some may wonder what manner of man was this who on technical grounds carried off a prize from a Newton. Through his patron, Tillotson, he obtained a living and was also made Head Master of the Free School at Enfield. He vacated the Fellowship by marriage, and took various scions of the nobility as boarders. Apparently he devoted too little time to the Free-placers, for the next we hear is that he was ejected by the Feoffees for neglecting the School. Nothing daunted he fought the case, won his suit, and was reinstated. His name comes down to us through his devotion to Botany. Among the exotics he imported was a new plant, which passed into the genus *Polymnia* of the Linnean system, retaining the trivial name of *Uvedalia*. His *Hortus Siccus* was sold for a large sum to Sir Robert Walpole. In the British Museum (Bibl. Sloan. Plut. 28 F.) are fifteen letters from Dr. Uvedale to Sir Hans Sloane.

We must not forget that to the license allowed to the Westminster boys whose playground appears to have been in part within the sacred walls of the Abbey itself, is due the lamentable incident recorded in the following extract from the *Gentleman's Magazine*, July, 1800. Anthony Sinnott, Gent. there records that on a recent visit he noticed "many statues and monuments injured, unreproved youth playing ball among them and throwing stones. . . ." The monument "erected over that luminary of science, Sir Isaac Newton, had two fingers broken from the left-hand of the figure."

At the risk of wearying some reader of the *Gazette*, we close with a few quotations that may be useful in class.

Is a man great because he is in advance of his age? Now that the welkin between Birmingham and Canterbury is ringing with controversy, a passage from a letter of Newton to Dr. Thos. Burnet is not untimely: "As to Moses, I do not think his description of y^e creation either philosophical or feigned, but that he described realities in a language artificially adapted to y^e sense of y^e vulgar . . . his business being not to correct the vulgar notions in matters philosophical, but to adapt a description of the creation as handsomely as he could to ye sense and capacity of y^e vulgar" (*v.* for the whole letter, Brewster, *Memoirs*, ii. pp. 447-454).

Or because he can in the instruction of others take broad views in preference to running the risk of being lost in details?

"Particular and contingent inventions in the solution of problems, which though many times more concise than a general method would allow, yet, in my judgment are less proper to instruct a learner, as acrostics, and such kind of artificial poetry, though never so excellent, would be but improper example to instruct one that aims at Ovidean poetry"—(to Collins, 1670).

Is modesty an essential?

"If I have done the public any service . . . it is due to nothing but industry and patient thought"—(to Bentley, 1692).

"If I have seen further than Descartes it is by standing on the shoulders of giants"—(to Hooke, 1676).

He could say of his theory of gravitation that his conviction of its truth increased tenfold from the moment he got one other person to believe in it.

"Go to De Moivre. He understands these things better than I do"—(a frequent reply to enquirers in his later years).

No wonder that a contemporary said that the approbation of Sir Isaac was more to him than the cry of all the ignorant.

Finally, here is Christopher North in one of the *Noctes Ambrosianae* (1856), vol. iii. p. 55.

"A monument to Newton! a monument to Shakespeare!

Look up to Heaven—look into the Human Heart!

Till the planets and the passions—the affections and the fixed stars are extinguished—their names cannot die!"

Perhaps we should now apologize to Dr. Brodetsky and to the reader for rambling at such length from the immediate subject of the review. But this stimulating little volume has aroused many memories, and they have carried us further than we anticipated when we began.

Union of South Africa List of Serials. New and revised edition. Compiled by A. C. G. Lloyd. Pp. 259. 1927. (Cape Town.)

Through the courtesy of Sir Thomas Muir, the above list has been placed in the Library of the Mathematical Association. The List includes the catalogue of his own mathematical library, a valuable collection which will eventually go to the South African Public Library. The 1350 serials of the 1921 list have increased to 3117 in 1927.

London University Guide. 1927-1928. (Burlington House, Cambridge.)

In this the University Correspondence College issues in compact form the information most likely to be useful to private and other students who are working for the examinations of the London University.

Pitman's Test Paper Series. Algebra. *With Answers and Points essential to Answers.* Second edition. By A. E. Donkin. Pp. 113+31a. 3s. 6d. 1927. *Junior.* Pp. 89. 1s. 3d. With Answers, 1s. 6d. **Arithmetic.** With Answers and Points, etc. Second edition. Pp. 119+31a. 3s. 6d. 1927. *Junior.* Pp. 80. 1s. 3d. With Answers, 1s. 6d. **Geometry.** By W. E. Paterson. Pp. 100. 2s. 1927. *Junior.* Pp. 75. 1s. 3d. 1927.

That Mr. Donkin's experience as Senior Mathematical Master at Rugby, and as Examiner for the Oxford, Cambridge, London and the Northern

Universities, has stood him in good stead, is indicated by the rapid appearance of a second edition of each of his collections of Test Papers. It will be remembered that the papers are for the use of School Certificate and Matriculation candidates. The "Points," which are a new feature in the second edition, are hints and suggestions which if closely followed will eliminate from the students' solutions as much drudgery as is possible. If used after a solution has been attempted, the "Points" will often be valuable to the private student in particular, especially in the selection of suitable units for x and y in the graphs.

Mr. Paterson, also an experienced examiner, has compiled fifty papers devised for candidates taking the examinations mentioned above. In an edition which has not reached us, "the answers to numerical questions and skeleton solutions to all but the simplest riders" are to be found.

We have received similar collections of papers in this series dealing with Physics (Senior and Junior), Mechanics and Physics, and Chemistry respectively.

The Junior Test Papers should be useful as preparatory to the Junior Locals, Lower Certificate and the like.

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483. Angle.—In 1844, one Upton—he does not deign to give his Christian name—published a somewhat curious volume entitled *Physioglyphics* (London, 8vo; Fisher, Son & Co.). At the conclusion of that work (p. 214) the author informs his readers that he is able "to give a practical way for trisecting any angle, or arc of a circle, and of such a nature that anyone may construct from it an instrument for the purpose." He further offers, if encouraged by a reasonable list of subscribers, to publish, at half-a-crown a copy, a "Practical Method for Trisecting an Angle, or Arc of a Circle." He also invites any three gentlemen to form themselves into a committee to investigate this method; stipulating, "in justice to the subscribers, that they keep the process a secret till the publication takes place; and that they give an immediate certificate, to satisfy the public, of the efficacy of the method, if convinced of it on *mathematical* principles." He further goes on to say, that though the method "will have to be *proved* on mathematical principles, it will not exactly develop the *mathematical process* by which he arrived at the solution"; and that if mathematicians "give it up," he will then "publish the mathematical process, which shall involve *no* assumption nor in *any respect deviate* from the *plain rules of Euclid*." This, he says in addition, will enable him to 'elucidate his views as to the existence of error," and that he will subsequently "follow them up by a *strict mathematical solution* of the QUADRATURE OF THE CIRCLE." Whether these promises were ever performed I do not know. Mathematicians do not believe in these angle-trisectors and circle-squarers, for the same reason that Coleridge disbelieved in ghosts—they have seen or heard of too many of them.—William Bates, *N. & Q.* IV. iii. p. 492, 1869.

484. . . . in the military school hard by, there sat, studying mathematics, a dusky complexioned, taciturn Boy, under the name of: NAPOLEON BONAPARTE!—Carlyle, *French Revolution*, Part I. Book III. Chap. VIII. [per L. L. Morgan, Rhodesia].

485. Condorcet, instructed by the Jesuits at Rheims, was able when he was only fifteen years old to go through such performances in analysis as to win special applause from illustrious judges like d'Alembert and Clairaut. It was impossible, however, for Jesuits, as it has ever been for all enemies of movement, to constrain within prescribed limits the activity which has once been effectively stirred. Mathematics has always been in the eyes of the Church a harmless branch of knowledge, but the mental energy that mathematics first touched is sure to turn itself by and by to more complex and dangerous subjects in the scientific hierarchy.—Morley, *Misc.* vol. ii. 1886, *Condorcet*, pp. 164, 167.

CORRESPONDENCE.

To the Editor of the *Mathematical Gazette.*

SIR,—I do not remember whether Mr. A. A. Milne studied mathematics at Cambridge, but anyone who has tried to explain that when we speak of the line at infinity we mean only the aggregate of the points at infinity on ordinary lines, and that when we say that P is the point at infinity on l we mean only that the line OP means the line through O parallel to l, will realise that the state of affairs is described very accurately in " Now We Are Six " :

" I think I am an Elephant
Behind another Elephant,
Behind *another* Elephant *who isn't really there.*"

Reading. E. H. NEVILLE.

CARDIFF BRANCH.

AT the Annual General Meeting of the Cardiff Branch on 24th October the following officers were elected for the session 1927-28 : President : Mr. A. Buxton, M.A., Technical College, Cardiff. Vice-Presidents : Dr. D. G. Taylor ; Emeritus Professor A. L. Selby ; Prof. H. R. Robinson ; Dr. J. H. Shaxby ; Mr. A. H. Pope ; Dr. R. T. Dunbar ; Mr. V. W. Evans. Hon. Secretary : Mr. H. A. Hayden, M.Sc., University College, Cardiff. Hon. Treasurer : Prof. G. H. Livens, M.A., University College, Cardiff. Committee : Mr. D. G. Reynolds ; Miss G. M. Weighill ; Mr. R. Langford ; Miss I. Viney.

YORKSHIRE BRANCH.

THE autumn meeting of the Yorkshire Branch of the Mathematical Association was held in the Mathematical Department of the University of Leeds on Saturday, 29th Oct. Ten new members were elected. A very satisfactory balance sheet was presented by Mr. W. Peaker, B.Sc. Professor S. Brodetsky, M.A., Ph.D., retired from the office of President of the Branch. He was very warmly thanked for all the time and energy he had put into the work of the Branch during his years of office. Mr. W. Peaker, B.Sc., of Cockburn High School, Leeds, was elected President in place of Professor Brodetsky.

Miss L. Watson, M.A., Girls' Modern School, Leeds, was elected Treasurer of the Branch. Miss Hooke, Miss Bowman, Mr. Ingham and Mr. Montagnon were elected members of committee. Mr. J. Smeal, M.Sc., of the University of Leeds, was elected auditor to the Branch.

The Secretary of the Branch, Mr. Arthur B. Oldfield (Pudsey), was presented with a fountain-pen, suitably inscribed, in recognition of his services in connection with the Sir Isaac Newton bicentenary celebrations held at Grantham last March.

Professor C. M. Jessop, M.A., Professor of Mathematics at Armstrong College, Newcastle-upon-Tyne, gave a most instructive paper on " The Principles of Line Geometry." Mr. A. Montagnon, M.A., Senior Mathematical Master at the Leeds Grammar School, also gave an excellent paper which was brimful of ideas and suggestions on " The Teaching of Mechanics in Schools."

ARTHUR B. OLDFIELD, *Hon. Sec.*

ERRATA.

P. 444. Figure. For S read v.

P. 447, l. 19 up. For F *read* E.

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